

G -symmetric spectra, semistability and the multiplicative norm

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Abstract

In this paper we develop the basic homotopy theory of G -symmetric spectra (that is, symmetric spectra with a G -action) for a finite group G , as a model for equivariant stable homotopy with respect to a G -set universe. This model lies in between Mandell's equivariant symmetric spectra and the G -orthogonal spectra of Mandell and May and is Quillen equivalent to the two. We further discuss equivariant semistability, construct model structures on module, algebra and commutative algebra categories and describe the homotopical properties of the multiplicative norm in this context.

0 Introduction

Stable equivariant homotopy theory has seen various applications both in equivariant and non-equivariant topology. Many of these applications make use of constructions which require or are simplified by a highly structured model of equivariant spectra with a point-set level smash product. For example, model structures on module or (commutative) algebra categories allow one to employ algebraic techniques in order to obtain new equivariant spectra with desired properties, fixed points of commutative equivariant ring spectra form non-equivariant E_∞ -ring spectra and their homotopy groups inherit the structure of a Tambara functor. Another example is the construction of the spectrum level multiplicative norm introduced in [HHR14], which plays a central role in the solution of the Kervaire invariant one problem.

The first highly structured models for stable equivariant homotopy theory were the S_G -modules and G -orthogonal spectra of [MM02] and the equivariant symmetric spectra of [Man04]. While the first two can be formed over arbitrary compact Lie groups, the latter only works for finite groups. In turn it has the advantage that it can also be based on simplicial sets, whereas the other two need topological spaces.

In this paper we develop a further model we call G -symmetric spectra. It also requires the group to be finite and can be based both on simplicial sets and topological spaces. Conceptually, it lies in between Mandell's equivariant symmetric spectra and G -orthogonal spectra. In order to describe the difference, we have to recall the definitions. G -orthogonal spectra are formed with respect to a G -representation universe (i.e., a countably infinite G -representation, which is isomorphic to a direct sum of two copies of itself), which determines what kinds of representation spheres S^V become invertible in the homotopy category. Then, roughly speaking, a G -orthogonal spectrum X consist of a family of based G -spaces $X(V)$ for every finite dimensional subrepresentation V of the chosen universe, and these are connected by structure maps of the form $X(V) \wedge S^W \rightarrow X(V \oplus W)$. In addition, every $X(V)$ possesses an action of the orthogonal group $O(V)$ which is suitably compatible with the G -action and the structure maps. G -orthogonal spectra formed with respect to different universes are connected by so-called change of universe functors. It was noticed in [MM02, V, Thm. 1.5] that on the point-set level these are always equivalences of categories (though the derived functors are not). In other words, the underlying category does not really depend on the chosen G -representation universe,

while the weak equivalences do. Furthermore, every such category of G -orthogonal spectra is equivalent to the category of G -objects in ordinary non-equivariant orthogonal spectra. In short, the reason for this phenomenon is that the values $X(V)$ can be reconstructed from those on trivial representations together with the orthogonal group actions. While this observation can be misleading in terms of the homotopy theory, it is very useful from a categorical viewpoint. For example, it is now clear how to define restriction to subgroups, induction, fixed point and orbit functors on the point set categories and it only remains to examine with respect to which model structures (or weak equivalences) they can be derived, in order to obtain corresponding functors between the homotopy categories. A further example for such a functor is the multiplicative norm of [HHR14] mentioned earlier in this introduction.

For Mandell's equivariant symmetric spectra (an equivariant version of the symmetric spectra of [HSS00]) the situation is somewhat different. There a normal subgroup N of G is fixed and a $G\Sigma_{G/N}$ -spectrum is defined as a family of based $(G \times \Sigma_n)$ -spaces (or -simplicial sets) $X(n \times G/N)$ for all natural numbers n , connected by structure maps of the form $X(n \times G/N) \wedge S^{G/N} \rightarrow X((n+1) \times G/N)$. The category of such spectra together with the model structure constructed in [Man04] is then Quillen equivalent to G -orthogonal spectra with respect to the universe of those G -representations on which N acts trivially (in particular it models the fully genuine theory in the case where N is the trivial subgroup). At first sight this looks similar to G -orthogonal spectra, replacing G -representations by G -sets and orthogonal groups by symmetric groups. But if N is a proper subgroup of G , then $X(n \times G/N)$ does not have an action of the full symmetric group of $n \times G/N$, but only of the subgroup of block permutations of the copies of G/N . Furthermore, it does not contain evaluations at trivial G -sets. Taken together, this results in the categories for varying N being pairwise non-equivalent, and only if N is equal to G the category of $G\Sigma_{G/N}$ -spectra is equivalent to the category of G -objects in non-equivariant symmetric spectra. As a consequence, to quote [Man04], “the construction of these [change of universe, change of groups, fixed point and orbit] functors and their relationship is significantly more complicated for equivariant symmetric spectra than it is for equivariant orthogonal spectra. In order to construct these functors we need to study ‘equivariant diagram spectra’ on diagrams more complicated than the one implicit in the definition of equivariant symmetric spectra.” The norm is not discussed in [Man04].

The G -symmetric spectra of this paper are the direct symmetric analog of G -orthogonal spectra. The category is always that of G -objects in the non-equivariant symmetric spectra introduced in [HSS00]. Parallel to the orthogonal case, these already secretly inherit evaluations at arbitrary finite G -sets. Which of these evaluations are declared homotopically meaningful in the model structure is based on a G -set universe \mathcal{U} , i.e., a countably infinite G -set which is isomorphic to the disjoint union of two copies of itself. The \mathbb{R} -linearization $\mathbb{R}[\mathcal{U}]$ of such a G -set universe is then a G -representation universe and we show in Theorem 5.1 that the model structures we construct are Quillen equivalent to the model structure of [MM02] on G -orthogonal spectra with respect to $\mathbb{R}[\mathcal{U}]$. It can happen that two non-isomorphic G -set universes become isomorphic when linearized and hence model the same homotopy theory. This leads to various different model structures with equivalent homotopy categories. For example, the N -fixed G -equivariant stable homotopy category can be modeled by a G -set universe consisting of infinitely many copies of the G -orbit G/N , which would be the closest approximation to [Man04]. But the disjoint union of infinitely many copies of all orbits G/H with $N \leq H$ leads to an equivalent theory and often has technical advantages, in particular for deriving fixed point functors or establishing multiplicative properties.

We now give a summary of the content of this paper. It is essentially self-contained and does not depend on the theory of G -orthogonal spectra, though - of course - a lot of arguments are similar to the ones in [MM02]. The fundamental comparison result is the following (where the references in brackets denote where it can be found in the paper):

Theorem A (Model structures, Theorems 4.10, 4.11, 5.1 and 5.3). *For every finite group G and every G -set universe \mathcal{U} there exists a model structure on G -symmetric spectra of spaces and simplicial sets, which is Quillen equivalent to the G -orthogonal spectra of [MM02] formed with respect to the G -representation universe $\mathbb{R}[\mathcal{U}]$. If \mathcal{U} is an infinite disjoint union of orbits G/N for a normal subgroup N of G , the model structure is also Quillen equivalent to the $G\Sigma_{G/N}$ -spectra of [Man04].*

The central notion is that of a $G^{\mathcal{U}}$ -stable equivalence (Definition 2.40), whose description is more complicated than in the orthogonal case. We will say more about this below in the paragraph on semistability. Around these $G^{\mathcal{U}}$ -stable equivalences we construct two kinds of model structures, which are useful for different purposes. The first one, called $G^{\mathcal{U}}$ -projective, is an equivariant generalization of the stable model structure in [HSS00] and analogous both to the stable model structures on G -orthogonal spectra constructed in [MM02] and the stable model structure on Mandell’s equivariant symmetric spectra in [Man04]. The second, called $G^{\mathcal{U}}$ -flat, is an equivariant generalization of the flat model structure on non-equivariant symmetric spectra constructed in [Shi04] and the analog of the model structures of [Sto11] on G -orthogonal spectra. The flat model structure has a lot more cofibrant objects than the projective one and the advantage that the cofibrations do not depend on the G -set universe \mathcal{U} . Furthermore, a map of G -symmetric spectra of simplicial sets is a G -flat cofibration if and only if it is a non-equivariant flat cofibration. We also construct positive versions of these model structures, which are used to obtain model structures on commutative G -symmetric ring spectra in Section 7. As it is standard for model structures on categories of spectra, in all cases we first construct a suitable level model structure (Section 2.9) which we then left Bousfield localize at the class of $G^{\mathcal{U}}$ -stable equivalences.

In Section 6 we explain how these model structures can be used to derive change of universe, change of groups, fixed point and orbit functors. Most of this discussion is similar to the one for G -orthogonal spectra ([MM02]) and rather straightforward. The only exception is that the flat model structure allows us to derive the change of universe functors for an inclusion of G -set universes $\mathcal{U} \subseteq \mathcal{U}'$ in the other direction than it is usually done (i.e., left deriving the identity from G -symmetric spectra with $G^{\mathcal{U}'}$ -stable equivalences to G -symmetric spectra with $G^{\mathcal{U}}$ -stable equivalences), but the resulting derived adjunction is equivalent to one we could have constructed more easily (Proposition 6.2). This also shows that the orbit functor can be left derived even when working over a non-trivial G -set universe, but again the resulting derived functor can be described more directly. In the case of a complete G -set universe it turns out that the derived orbits are equivalent to the homotopy orbits (Proposition 6.15).

Another emphasis of this paper is the theory of equivariant semistability: In Section 3 we define the equivariant homotopy groups of a G -symmetric spectrum, depending on the G -set universe \mathcal{U} , and show that (just like for G -orthogonal spectra) they carry a natural structure of a $G^{\mathcal{U}}$ -Mackey functor. Here, the choice of G -set universe \mathcal{U} has an impact on the functoriality (cf. [Lew95]). However, unlike for G -orthogonal spectra, a $G^{\mathcal{U}}$ -stable equivalence does not necessarily induce an isomorphism on these “naively defined” equivariant homotopy groups (it is not automatically a $\pi_*^{\mathcal{U}}$ -isomorphism), i.e., they are not homotopical and need to be (right) derived. This problem is already present non-equivariantly and constitutes the main technical difference to orthogonal spectra. In [HSS00] and in particular in [Sch08], various criteria are developed to detect whether a (non-equivariant) symmetric spectrum has the correct naive homotopy groups, i.e., for which symmetric spectra the comparison map to the derived homotopy groups (those of an Ω -spectrum replacement) is an isomorphism. Such symmetric spectra are called semistable and are often better behaved. For example, for maps between semistable symmetric spectra the classes of stable equivalences and π_* -isomorphisms coincide. A conceptual explanation of many of the phenomena related to non-equivariant semistability is given in [Sch08], where Schwede shows that the naive homotopy groups of a symmetric spectrum carry a natural (“tame”) action of the monoid of injective self-maps of the natural numbers,

and that a symmetric spectrum is semistable if and only if this action is trivial.

This generalizes equivariantly, for which we likewise call a G -symmetric spectrum $G^{\mathcal{U}}$ -semistable if the comparison map from its naive equivariant homotopy groups to the derived ones is an isomorphism:

Theorem B (Semistability, Theorem 3.47). *The naive equivariant homotopy groups of a G -symmetric spectrum carry a natural, additive and tame action of the monoid of G -equivariant injective self-maps of the G -set universe \mathcal{U} , compatible with the $G^{\mathcal{U}}$ -Mackey functor structure. A G -symmetric spectrum is $G^{\mathcal{U}}$ -semistable if and only if this action is trivial. Furthermore, the class of $G^{\mathcal{U}}$ -semistable G -symmetric spectra is closed under wedges, shifts, cones, smash products (as long as one factor is also G -flat), $\text{map}(A, -)$ for a finite based G -CW complex A , restriction to a subgroup and both induction and coinduction from a subgroup. It contains all $G^{\mathcal{U}}\Omega$ -spectra, restrictions of G -orthogonal spectra (in particular, suspension spectra) and all G -symmetric spectra X for which every naive homotopy group $\pi_n^{H, \mathcal{U}} X$ is finitely generated.*

In the paper we proceed the other way around and take the triviality of the monoid action as the definition for $G^{\mathcal{U}}$ -semistable G -symmetric spectra. Generalizing non-equivariant results of [HSS00] and [Sch08], other criteria for $G^{\mathcal{U}}$ -semistability are given in Section 3, where the behavior of naive equivariant homotopy groups under various constructions is examined, together with their monoid action.

While a $G^{\mathcal{U}}$ -stable equivalence does not necessarily induce isomorphisms on naive equivariant homotopy groups, the opposite is true. As in the non-equivariant case this not obvious from the definition, but can be shown similarly as in [HSS00].

Theorem C (Theorem 3.48). *Every $\pi_*^{\mathcal{U}}$ -isomorphism of G -symmetric spectra is a $G^{\mathcal{U}}$ -stable equivalence.*

This can be used to show that for maps between $G^{\mathcal{U}}$ -semistable G -symmetric spectra the notions of $\pi_*^{\mathcal{U}}$ -isomorphism and $G^{\mathcal{U}}$ -stable equivalence agree (Corollary 3.49).

Multiplicative properties of the projective and flat model structures are discussed in Section 7. We first show, using the machinery of [SS00]:

Theorem D (Model structures on modules and algebras, Corollary 7.4).

- (i) *For every G -symmetric ring spectrum R the positive and non-positive, stable $G^{\mathcal{U}}$ -flat and $G^{\mathcal{U}}$ -projective model structures lift to the category of R -modules. If R is commutative, they are again monoidal.*
- (ii) *For every commutative G -symmetric ring spectrum R the positive and non-positive, stable $G^{\mathcal{U}}$ -flat and $G^{\mathcal{U}}$ -projective model structures lift to the category of R -algebras.*

In all the above model structures, a map is a weak equivalence (or fibration) if and only if the underlying map of G -symmetric spectra is one in the respective model structure.

To apply [SS00] we show that the monoid axiom ([SS00, Def. 3.3]) is satisfied for all these model structures (Proposition 7.3). It is implied by Proposition 7.1, which says that smashing with a G -flat G -symmetric spectrum preserves all $G^{\mathcal{U}}$ -stable equivalences and not only those between G -flat G -symmetric spectra.

Furthermore, we consider the multiplicative norm of [HHR14], a multiplicative version of induction from H -symmetric spectra to G -symmetric spectra for a subgroup H of G . While the rest of the paper works with respect to arbitrary G -set universes, we here restrict to the full N -fixed ones with respect to a normal subgroup N of G , denoted $\mathcal{U}_G(N)$. We show:

Theorem E (Homotopical properties of the norm, Theorem 7.8). *Let $N \leq H \leq G$ be subgroups with N normal in G . Then the norm functor $N_H^G : HSp^{\Sigma} \rightarrow GSp^{\Sigma}$ takes $H^{\mathcal{U}_H(N)}$ -stable equivalences between H -flat H -symmetric spectra to $G^{\mathcal{U}_G(N)}$ -stable equivalences of G -flat G -symmetric spectra.*

Hence, the norm functor can be left derived with respect to these weak equivalences. Since every $H^{\mathcal{U}_H(N)}$ -projective cofibration is also an H -flat cofibration, this theorem in particular applies to maps between $H^{\mathcal{U}_H(N)}$ -projective H -symmetric spectra. In fact, over the projective model structures a stronger result holds (Theorem 7.7).

Theorem E also plays a role in the construction of model structures on categories of commutative algebras. As in the non-equivariant case, only the positive versions lift:

Theorem F (Model structures on commutative algebras, Theorem 7.13). *Let R be a commutative G -symmetric ring spectrum and N a normal subgroup of G . Then the positive $G^{\mathcal{U}_G(N)}$ -projective and the positive $G^{\mathcal{U}_G(N)}$ -flat model structures lift to the category of commutative R -algebras.*

Again, the weak equivalences and fibrations are created in the underlying G -symmetric spectra. In establishing these model structures we make use of results of [Whi14]. Non-equivariantly, a major step in the proof is that given a positive flat symmetric spectrum X , the maps $(E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \rightarrow X^{\wedge n}/\Sigma_n$, collapsing $E\Sigma_n$ to a point, are stable equivalences for all n . This is also the case equivariantly, but the G -equivariant generalization of $E\Sigma_n$ which is needed depends on the universe with regard to which the stable equivalences are formed (Proposition 7.18). When working over $\mathcal{U}_G(N)$, it is the universal principal Σ_n -bundle in the category of G -spaces with trivial N -action, or equivalently a universal space for the family of subgroups of $G \times \Sigma_n$ which are of the form $\{(h, \varphi(h)) \mid h \in H\}$ for a subgroup H of G and a group homomorphism $\varphi : H \rightarrow \Sigma_n$ with $H \cap N$ in the kernel. Only if the G -set universe is trivial one can use an $E\Sigma_n$ with trivial G -action. As remarked in [HHR14], this was overlooked in [MM02] (and also in [Man04]). It is the appearance of the $E\Sigma_n$'s with non-trivial G -action which connects this problem to the homotopical properties of the norm.

The class of positive $G^{\mathcal{U}_G(N)}$ -flat cofibrations of commutative R -algebras again does not depend on the chosen universe $\mathcal{U}_G(N)$ and hence we simply call them positive G -flat cofibrations. They satisfy the “convenience” property of their non-equivariant version discussed in [Shi04]:

Theorem G (Convenience, Proposition 7.14). *The underlying R -module map of a positive G -flat cofibration of commutative R -algebras $X \rightarrow Y$ is a positive G -flat cofibration of R -modules if X is (not necessarily positive) G -flat as an R -module. In particular, the G -symmetric spectrum underlying a positive G -flat commutative G -symmetric ring spectrum is positive G -flat.*

As in the non-equivariant case, the projective analog of this result does not hold. Since the norm functor is (strong) symmetric monoidal, it maps commutative H -symmetric ring spectra to commutative G -symmetric ring spectra. The resulting functor is left adjoint to the forgetful functor. Since both weak equivalences and fibrations are created in the underlying category of equivariant symmetric spectra, it follows directly from the change of groups results that this adjunction becomes a Quillen pair, both with respect to the flat and the projective model structures. Moreover, the underlying G -symmetric spectrum of the derived norm of a commutative H -symmetric ring spectrum is equivalent to the derived norm of the underlying H -symmetric spectrum (Corollary 7.15). This is not a priori clear, but it follows from the convenience of the flat model structures together with Theorem E.

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1 G -spaces and G -simplicial sets

1.1 Elementary definitions and notation

Let G be a finite group. Throughout this paper a topological space means a compactly generated weak Hausdorff space, so that the category of topological spaces becomes closed symmetric monoidal with respect to the cartesian product.

Definition 1.1. A based G -space is a topological space with an action of G through homeomorphisms, together with a chosen basepoint which is fixed under the action of G . A G -map between based G -spaces is a continuous based map which commutes with the given G -actions. We denote the category of based G -spaces together with based G -maps by $G\mathbf{T}_*$.

Definition 1.2. Likewise, a based G -simplicial set is a based simplicial set with a G -action or equivalently a functor $\Delta^{op} \rightarrow G\text{-sets}_*$. The category of based G -simplicial sets is denoted $G\mathcal{S}_*$.

Definition 1.3. For a subgroup H of G and a based G -space/ G -simplicial set X we denote the based space/simplicial set of H -fixed points of X (which are taken degreewise in the simplicial case) by X^H . The H -orbits X/H are defined to be the quotient by the H -action.

In the case where H is a normal subgroup, both X^H and X/H carry an induced action of the quotient group G/H .

Definition 1.4. The geometric realization functor $G\mathcal{S}_* \rightarrow G\mathbf{T}_*$ is denoted by $|\cdot|$ and its right adjoint singular complex $G\mathbf{T}_* \rightarrow G\mathcal{S}_*$ by \mathcal{S} .

It follows from adjointness that $|\cdot|$ commutes with G -orbits and \mathcal{S} commutes with G -fixed points. Moreover, since $|\cdot|$ commutes with all finite limits, it also preserves G -fixed points.

Definition 1.5. The smash product $X \wedge Y$ of two based G -spaces/ G -simplicial sets X and Y is the cofiber of the canonical map $X \vee Y \rightarrow X \times Y$. If not specified otherwise, it is equipped with the diagonal G -action.

The canonical G -homeomorphism $|X \times Y| \cong |X| \times |Y|$ descends to a G -homeomorphism $|X \wedge Y| \cong |X| \wedge |Y|$ for any pair of based G -simplicial sets X and Y ,

Example 1.6 (Representation spheres). Let G be a finite group and V a finite dimensional G -representation, i.e., a finite dimensional real vector space equipped with a scalar product and a G -action by linear isometries. Then the one-point compactification of V with induced G -action is called the *representation sphere* of V and denoted S^V . It is based at infinity. For another G -representation W there is a canonical G -homeomorphism $S^{V \oplus W} \cong S^V \wedge S^W$ sending a pair (v, w) to $v \wedge w$ and the basepoint to the basepoint.

A special case of this construction, which is of particular importance to us, is the representation sphere associated to a finite G -set M , namely $S^M := S^{\mathbb{R}[M]}$ where $\mathbb{R}[M]$ is the \mathbb{R} -linearization of M for which the canonical basis is orthonormal. These can also be defined over simplicial sets: Taking S^1 to be the quotient $\underline{\Delta}^1 / \partial \underline{\Delta}^1$ (where $\underline{\Delta}^1$ denotes the standard 1-simplex), based at $\partial \underline{\Delta}^1$, we set $S^M := (S^1)^{\wedge M}$, with G permuting the smash factors according to its action on M . Fixing a based homeomorphism $|S^1| \cong S^1 = S^{\mathbb{R}}$ induces a G -homeomorphism $|S^M| = |(S^1)^{\wedge M}| \cong |S^1|^{\wedge M} \cong (S^1)^{\wedge M} \cong S^M$ for all finite G -sets M .

Definition 1.7 (Mapping spaces). The *mapping space* $\text{map}(X, Y)$ of two based G -spaces X and Y is the set of all continuous basepoint preserving maps from X to Y equipped with the compact-open topology. The space $\text{map}(X, Y)$ carries a G -action via conjugation, the G -fixed points are given by the subspace of G -equivariant maps $\text{map}_G(X, Y)$.

For two based G -simplicial sets X and Y the simplicial mapping space $\text{map}(X, Y)$ has as n -simplices the set of all based simplicial maps $(\underline{\Delta}^n)_+ \wedge X \rightarrow Y$ with faces and degeneracies through

the cosimplicial object $\underline{\Delta}$ in the first variable. Again, this carries a G -action via conjugation, where G acts trivially on $(\underline{\Delta}^n)_+$.

Remark 1.8. The definitions above give two enrichments of $G\mathbf{T}_*$, one over $G\mathbf{T}_*$ itself with mapping spaces $\text{map}(-, -)$ with conjugation G -action and one over \mathbf{T}_* using $\text{map}_G(-, -)$. Likewise, $G\mathcal{S}_*$ is enriched over $G\mathcal{S}_*$ and \mathcal{S}_* .

Remark 1.9. The smash product together with the mapping spaces $\text{map}(-, -)$ make $G\mathbf{T}_*$ and $G\mathcal{S}_*$ closed symmetric monoidal categories with monoidal unit S^0 .

Definition 1.10 (Change of groups). A subgroup inclusion $H \rightarrow G$ induces *restriction* functors $\text{res}_H^G : G\mathbf{T}_* \rightarrow H\mathbf{T}_*$ and $\text{res}_H^G : G\mathcal{S}_* \rightarrow H\mathcal{S}_*$ by sending a based G -space/ G -simplicial set X to the same space/simplicial set with restricted H -action and not changing the morphisms.

The restriction functor has both a left and a right adjoint. The prior is given by *induction* $G \ltimes_H - : H\mathbf{T}_* \rightarrow G\mathbf{T}_*$ (respectively $G \ltimes_H - : H\mathcal{S}_* \rightarrow G\mathcal{S}_*$), sending a based H -space/ H -simplicial set X to the based space/simplicial set $G \ltimes_H X := G_+ \wedge_H X$ with G -action via $g[g', x] = [gg', x]$. The right adjoint is called *coinduction* and given by the space $\text{map}_H(G_+, X)$ of H -equivariant maps from G to X , with G -action through $(g \cdot \varphi)(g') = \varphi(g'g)$.

If one composes restriction with induction, one obtains a decomposition, called the *double coset formula*, which will be used throughout this paper. Given a finite group G and two subgroups H and K , we let $K \backslash G / H$ denote the set of double cosets, i.e., equivalence classes of G under the $K \times H$ action given by $(k, h) \cdot g = kgh^{-1}$. Then for every based H -space/ H -simplicial set X there is a natural K -equivariant decomposition

$$\text{res}_K^G(G \ltimes_H X) \cong \bigvee_{[g] \in K \backslash G / H} K \ltimes_{K \cap gHg^{-1}} (c_g^*(\text{res}_{g^{-1}Kg \cap H}^G X)).$$

Here, c_g^* stands for restriction along the conjugation isomorphism $g^{-1}(-)g : K \cap gHg^{-1} \rightarrow g^{-1}Kg \cap H$. On the summand belonging to a double coset representative g , the isomorphism above is the adjoint of the $K \cap gHg^{-1}$ -equivariant map $c_g^*(\text{res}_{g^{-1}Kg \cap H}^G X) \rightarrow \text{res}_{K \cap gHg^{-1}}^G X$ given by multiplication with g .

Likewise, there is the following decomposition of coinduction followed by restriction:

$$\text{res}_K^G(\text{map}_H(G, X)) \cong \prod_{[g] \in K \backslash G / H} \text{map}_{K \cap gHg^{-1}}(K, c_g^*(\text{res}_{g^{-1}Kg \cap H}^G X)).$$

Furthermore, induction and coinduction are related by a natural transformation $\gamma_X : G \ltimes_H X \rightarrow \text{map}_H(G_+, X)$, which is adjoint to the H -map $X \rightarrow \text{map}_H(G_+, X)$ sending x to the function which maps g to gx if g is contained in H and the basepoint otherwise.

Definition 1.11. A *relative G -CW complex* is a pair of G -spaces (X, A) equipped with a filtration $A = X_{-1} \subseteq X_0 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq X$ such that each X_n can be obtained from X_{n-1} by forming the pushout of the inclusion $(T \times S^{n-1})_+ \rightarrow (T \times D^n)_+$ along a map $(T \times S^{n-1})_+ \rightarrow X_{n-1}$ (for some G -set T and trivial G -action on S^{n-1} and D^n) and such that X is the colimit of the X_i .

A *based G -CW complex* is a relative G -CW complex of the form $(X, *)$. The geometric realization of any based G -simplicial set comes equipped with a natural G -CW structure via the realizations of its skeleta. Furthermore, both the G -orbits X/G and the G -fixed points X^G of a based G -CW complex X have an induced structure of a non-equivariant CW complex.

Example 1.12. All representation spheres S^V for a finite dimensional G -representation V can be equipped with the structure of a G -CW complex. For linearizations of finite G -sets this follows from the simplicial description above, for general V it is a consequence of a result of Illman [Ill78].

Definition 1.13. Two G -maps $f, g : X \rightarrow Y$ of based G -spaces/ G -simplicial sets are said to be based G -homotopic if they lie in the same path component of $\text{map}_G(X, Y)$. We denote the set of G -homotopy classes of based G -maps by $[X, Y]^G$.

1.2 Genuine model structure

We review certain model structures on based G -spaces and based G -simplicial sets, starting with the genuine model structure. For the notions of non-equivariant Serre and Kan fibrations and model category terminology we refer to the textbooks [Qui67], [Hir03] or [Hov99].

Definition 1.14. A map $f : X \rightarrow Y$ of based G -spaces (based G -simplicial sets) is a

- *genuine G -equivalence* if for all subgroups H of G the fixed point map $f^H : X^H \rightarrow Y^H$ is a weak homotopy equivalence of spaces (simplicial sets).
- *genuine G -fibration* if for all subgroups H of G the fixed point map $f^H : X^H \rightarrow Y^H$ is a Serre fibration of spaces (Kan fibration of simplicial sets).
- *genuine G -cofibration* if it has the left lifting property with respect to all genuine G -fibrations that are also genuine G -equivalences.

In the simplicial case a map is a genuine G -cofibration if and only if it is levelwise injective.

Proposition 1.15 (Genuine model structure). *The classes of genuine G -equivalences, genuine G -fibrations and genuine G -cofibrations define a proper, cofibrantly generated and monoidal model structure on the category of based G -spaces (based G -simplicial sets). Sets of generating cofibrations and generating acyclic cofibrations are given by*

$$\begin{aligned} I_G &:= \{(G/H \times S^{n-1})_+ \rightarrow (G/H \times D^n)_+ \mid H \leq G, n \in \mathbb{N}\} \\ J_G &:= \{(G/H \times D^n)_+ \rightarrow (G/H \times D^n \times [0, 1])_+ \mid H \leq G, n \in \mathbb{N}\} \end{aligned}$$

in the case of based G -spaces and

$$\begin{aligned} I_G &:= \{(G/H \times \partial \underline{\Delta}^n)_+ \rightarrow (G/H \times \underline{\Delta}^n)_+ \mid H \leq G, n \in \mathbb{N}\} \\ J_G &:= \{(G/H \times \Lambda_k^n)_+ \rightarrow (G/H \times \underline{\Delta}^n)_+ \mid H \leq G, n \geq 1, 0 \leq k \leq n \in \mathbb{N}\} \end{aligned}$$

in the case of based G -simplicial sets.

The genuine model structures are well-known. In the topological case a reference is [MM02, III.1.8].

Throughout the paper we will use the expression *cofibrant G -space* to mean a based topological G -space or G -simplicial set which is cofibrant in the genuine model structure above. These are precisely the retracts of I_G -cell complexes. Every G -simplicial set is cofibrant. Furthermore, by a *finite* cofibrant G -space we will mean a retract of a finite I_G -cell complex. For a G -simplicial set this is equivalent to having only finitely many non-degenerate simplices.

1.3 Model structures with respect to a family of subgroups

The later treatment of spectra will show the need for other model structures on G -spaces and G -simplicial sets where weak equivalences are those G -maps that induce weak homotopy equivalences only on fixed points for a chosen set of subgroups of G , more precisely for a *family* of subgroups.

Definition 1.16. Let G be a finite group. A non-empty set of subgroups of G is called a *family* if it is closed under conjugation and passage to subgroups.

Families of the following type are of particular importance to us:

Example 1.17. Let G and K be two finite groups. Then the family of subgroups of $G \times K$ which intersect $1 \times K$ trivially is denoted by $\mathcal{F}^{G,K}$. Every such subgroup is of the form $\{(h, \varphi(h)) \mid h \in H\}$ for a unique subgroup H of G and group homomorphism $\varphi : H \rightarrow K$.

To each family one can associate a universal G -space $E\mathcal{F}$, i.e., an (unbased) G -CW complex with the property that the H -fixed points for a subgroup H are contractible if H is in \mathcal{F} and empty otherwise. Such a universal space always exists and is unique up to G -homotopy equivalence. We call a G -simplicial set universal for \mathcal{F} (and also denote it $E\mathcal{F}$) if its geometric realization is a universal space for \mathcal{F} .

Definition 1.18 (\mathcal{F} -equivalence). A G -map $f : X \rightarrow Y$ of based G -spaces or based G -simplicial sets is called an \mathcal{F} -equivalence if for all subgroups H of G that lie in \mathcal{F} the fixed point map $f^H : X^H \rightarrow Y^H$ is a weak homotopy equivalence, or equivalently if $E\mathcal{F}_+ \wedge f$ is a genuine G -equivalence.

We will need the following two versions of the equivariant Whitehead theorem: (cf. [Ada84, Prop. 2.7])

Proposition 1.19 (Whitehead theorem). • *Let X be a G -CW complex with isotropy in \mathcal{F} and $f : Y \rightarrow Z$ an \mathcal{F} -equivalence. Then f induces a bijection $[X, Y]^G \cong [X, Z]^G$.*

- *Let (X, A) be a relative G -CW complex with all relative isotropy in \mathcal{F} and Z a based G -space which is \mathcal{F} -equivalent to a point. Then the restriction map $[X, Z]^G \rightarrow [A, Z]^G$ is a bijection.*

Using this one obtains that a based G -map $f : X \rightarrow Y$ of G -spaces is an \mathcal{F} -equivalence if and only if $\text{map}(E\mathcal{F}_+, f)$ is a genuine G -equivalence. We say that a based G -space X is \mathcal{F} -local if the canonical map $X \rightarrow \text{map}(E\mathcal{F}_+, X)$ is a genuine G -equivalence. By 2-out-of-3 for genuine G -equivalences and the previous lemma, it follows that a map between \mathcal{F} -local G -spaces is an \mathcal{F} -equivalence if and only if it is a genuine G -equivalence. A based G -simplicial set X is said to be \mathcal{F} -local if its geometric realization is.

The class of \mathcal{F} -equivalences take part in two model structures. For this we define:

Definition 1.20. A map $f : X \rightarrow Y$ of based G -spaces/ G -simplicial sets is called a

- *projective \mathcal{F} -cofibration* if it is a genuine G -cofibration where all the points away from the image of f have isotropy in \mathcal{F} .
- *projective \mathcal{F} -fibration* if it induces a Serre/Kan fibration $f^H : X^H \rightarrow Y^H$ for all $H \in \mathcal{F}$.
- *mixed \mathcal{F} -fibration* if it has the right lifting property with respect to all maps that are genuine cofibrations and \mathcal{F} -equivalences.

Proposition 1.21 (Projective model structure). *Let G be a finite group and \mathcal{F} a family of subgroups. Then the classes of projective \mathcal{F} -cofibrations, \mathcal{F} -equivalences and projective \mathcal{F} -fibrations assemble to a proper, cofibrantly generated, monoidal and G -topological (G -simplicial) model structure on based G -spaces (G -simplicial sets), called the projective \mathcal{F} -model structure.*

Here and below, G -topological or G -simplicial is meant with respect to the genuine model structure of Proposition 1.15. The projective model structure is cofibrantly generated by the subsets $I_{G, \text{proj}}^{\mathcal{F}}$ of I_G and $J_{G, \text{proj}}^{\mathcal{F}}$ of J_G of those inclusions with relative isotropy in \mathcal{F} .

Proposition 1.22 (Mixed model structure). *Let G be a finite group and \mathcal{F} a family of subgroups. Then the classes of genuine G -cofibrations, \mathcal{F} -equivalences and mixed \mathcal{F} -fibrations assemble to a proper, cofibrantly generated, monoidal and G -topological (G -simplicial) model structure on based G -spaces (G -simplicial sets), called the mixed \mathcal{F} -model structure.*

The mixed model structure can be obtained from the genuine model structure by applying Bousfield's localization theorem [Bou01, Theorem 9.3] with respect to the replacement $\alpha_X : X \rightarrow \text{map}(E\mathcal{F}_+, X^f)$, where $(-)^f$ is a genuine fibrant replacement functor (e.g. the identity for based G -spaces and $\mathcal{S}(|\cdot|)$ for based G -simplicial sets). It then follows that mixed \mathcal{F} -fibrations can be characterized as exactly those based G -maps $f : X \rightarrow Y$ which are genuine G -fibrations and for which in addition the square

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & \text{map}(E\mathcal{F}_+, X^f) \\ f \downarrow & & \downarrow \text{map}(E\mathcal{F}_+, f^f) \\ Y & \xrightarrow{\alpha_Y} & \text{map}(E\mathcal{F}_+, Y^f) \end{array}$$

is homotopy cartesian in the genuine model structure. In particular, a based G -space/ G -simplicial set is mixed \mathcal{F} -fibrant if and only if it is genuinely G -fibrant and \mathcal{F} -local. Using this characterization of the fibrations and the adjunction between smash product and mapping space, we see that the mixed model structure is cofibrantly generated with the previous I_G as generating cofibrations and

$$J_{G, \text{mix}}^{\mathcal{F}} := \{j \square \alpha \mid j \in J_G\} \cup J_G$$

as generating acyclic cofibrations. Here, α is the inclusion $E\mathcal{F}_+ \rightarrow C(E\mathcal{F})_+$ into the cone and \square denotes the pushout-product of two maps.

2 G -symmetric spectra

In this section we introduce G -symmetric spectra, explain various point-set level constructions and construct level model structures.

2.1 Definition

Definition 2.1 (Symmetric spectrum). A *symmetric spectrum* X of spaces or simplicial sets consists of

- a based Σ_n -space/ Σ_n -simplicial set X_n and
- a based *structure-map* $\sigma_n : X_n \wedge S^1 \rightarrow X_{n+1}$

for all $n \in \mathbb{N}$. This data has to satisfy the condition that for all $n, m \in \mathbb{N}$ the *iterated structure map*

$$\sigma_n^m : X_n \wedge S^m \cong (X_n \wedge S^1) \wedge S^{m-1} \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \rightarrow \dots \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$ -equivariant. Here, the $\Sigma_n \times \Sigma_m$ -action on $X_n \wedge S^m$ is the smash product of the Σ_n -action on X_n and the Σ_m -action on S^m as the permutation sphere for the natural Σ_m -set $\underline{m} := \{1, \dots, m\}$. The action on X_{n+m} is induced from the inclusion $\Sigma_n \times \Sigma_m \rightarrow \Sigma_{n+m}$ which arises from disjoint union and the bijection $\underline{n} \sqcup \underline{m} \cong \underline{n+m}$ that sends each k in \underline{n} to itself and each l in \underline{m} to $n+l$.

A *map of symmetric spectra* $f : X \rightarrow Y$ is a sequence of based Σ_n -equivariant maps $f_n : X_n \rightarrow Y_n$ such that $f_{n+1} \circ \sigma_n^{(X)} = \sigma_{n+1}^{(Y)} \circ (f_n \wedge S^1)$ for all $n \in \mathbb{N}$.

We denote the category of symmetric spectra over spaces or simplicial sets by $Sp_{\mathbf{T}}^{\Sigma}$ or $Sp_{\mathbf{S}}^{\Sigma}$, respectively. In statements that make sense over both categories we sometimes simply write Sp^{Σ} and mean they hold in either of them.

Definition 2.2 (*G*-symmetric spectrum). Let G be a finite group. A *G*-symmetric spectrum of spaces or simplicial sets is a symmetric spectrum together with a G -action via automorphisms of symmetric spectra, or equivalently a functor $G \rightarrow Sp_{\mathbf{T}}^{\Sigma}$ or $G \rightarrow Sp_{\mathbf{S}}^{\Sigma}$. A *map of G-symmetric spectra* is a map of symmetric spectra that commutes with the given G -actions. We denote the categories of G -symmetric spectra by $GSp_{\mathbf{T}}^{\Sigma}$ and $GSp_{\mathbf{S}}^{\Sigma}$.

Equivalently, a G -symmetric spectrum is a symmetric spectrum X together with a G -action on each level X_n which commutes with the Σ_n -action and for which all structure maps $\sigma_n : X_n \wedge S^1 \rightarrow X_{n+1}$ are G -equivariant for the trivial action on S^1 .

Example 2.3 (Suspension spectra). Let A be a based G -space (G -simplicial set). Then the *suspension spectrum* of A , denoted $\Sigma^{\infty}A$, has $(\Sigma^{\infty}A)_n := A \wedge S^n$ as its n -th level where the $(G \times \Sigma_n)$ -action is the smash product of the G -action on A and the permutation Σ_n -action on S^n . The structure map is the associativity homeomorphism

$$(A \wedge S^n) \wedge S^1 \cong A \wedge (S^n \wedge S^1) \cong A \wedge S^{n+1}.$$

Taking A to be S^0 with trivial G -action yields the sphere spectrum $\Sigma^{\infty}S^0$, which we also denote by \mathbb{S} .

The categories $GSp_{\mathbf{T}}^{\Sigma}$ and $GSp_{\mathbf{S}}^{\Sigma}$ have all limits and colimits and these are computed levelwise. More precisely, for I a small category and $F : I \rightarrow Sp^{\Sigma}$ a functor the G -symmetric spectrum $\text{colim } F$ with $(\text{colim } F)_n := \text{colim}(F_n)$ and structure map

$$\text{colim}(F_n) \wedge S^1 \cong \text{colim}(F_n \wedge S^1) \xrightarrow{\text{colim}(\sigma_n)} \text{colim}(F_{n+1})$$

is a colimit for F . A limit is given by $\lim F$ with $(\lim F)_n := \lim(F_n)$ and structure map adjoint to

$$\lim(F_n) \xrightarrow{\lim(\tilde{\sigma}_n)} \lim(\Omega F_{n+1}) \cong \Omega \lim(F_{n+1}).$$

2.2 Evaluation on finite G -sets and generalized structure maps

For the homotopy theory of G -symmetric spectra it is essential that they can be evaluated on finite G -sets, which we now explain.

Let M be a finite G -set of order m . We denote by $\text{Bij}(\underline{m}, M)$ the discrete space or simplicial set of bijections between the sets $\underline{m} = \{1, \dots, m\}$ and M . It possesses a right Σ_m -action by precomposition and a left Σ_M -action by postcomposition.

Definition 2.4 (Evaluation). The *evaluation* of a G -symmetric spectrum X on a finite G -set M is defined by

$$\begin{aligned} X(M) &:= X_m \wedge_{\Sigma_m} \text{Bij}(\underline{m}, M)_+ \\ &:= X_m \wedge \text{Bij}(\underline{m}, M)_+ / \{(\sigma x, f) \sim (x, f\sigma), \sigma \in \Sigma_m\} \end{aligned}$$

with diagonal G -action $g[x \wedge f] := [gx \wedge gf]$.

Remark 2.5. The notation $X_m \wedge_{\Sigma_m} \text{Bij}(\underline{m}, M)_+$ might be slightly confusing since Σ_m acts from the left on X_m and from the right on $\text{Bij}(\underline{m}, M)_+$. In analogy to the tensor product of modules one would expect the notation $\text{Bij}(\underline{m}, M)_+ \wedge_{\Sigma_m} X_m$. We write it the other way around because both this construction and the structure maps $X_m \wedge S^1 \rightarrow X_{m+1}$ are special cases of the action of the category $G\Sigma$ on the levels of a G -symmetric spectrum, as we describe in Section 2.7, and we have chosen to define sphere actions from the right.

The underlying space/simplicial set of $X(M)$ should be thought of as a coordinate change. Since $\text{Bij}(\underline{m}, M)$ is free and transitive as a right Σ_m -set, any choice of bijection from \underline{m} to M gives rise to a non-equivariant isomorphism from X_m to $X(M)$. In particular, $X(\underline{m})$ is canonically isomorphic to X_m . However, the G -action on $X(M)$ is usually different from that on X_m . It is twisted by the G -set structure on M , which corresponds to a group homomorphism from G into Σ_M . In particular, the G -action on $X(M)$ depends on the Σ_m -action on X_m .

Evaluation is functorial in isomorphisms of finite sets. Given any bijective map $\varphi : M \rightarrow N$ we obtain an induced isomorphism $X(\varphi) : X(M) \rightarrow X(N)$ by sending $[x \wedge f]$ to $[x \wedge (\varphi \circ f)]$. This isomorphism is in general only G -equivariant if φ is. In particular, the induced Σ_M -action on $X(M)$ does in general not commute with the G -action. However, the two assemble to an action by the semi-direct product $G \ltimes \Sigma_M$ associated to the composition $G \rightarrow \Sigma_M \xrightarrow{\text{conj}} \text{Aut}(\Sigma_M)$, where the first map is the group homomorphism defining the G -set structure on M .

The following are two examples of evaluations:

Example 2.6. Let A be a based G -space or G -simplicial set, M a finite G -set. Then the map $(\Sigma^\infty A)(M) \rightarrow A \wedge S^M$ that sends a class $[(a \wedge x) \wedge f]$ to $a \wedge f_*(x)$ is a G -isomorphism for the diagonal action on the target.

We note that for non-trivial M the G -action on $\mathbb{S}(M) \cong S^M$ is non-trivial even though the one on the sphere spectrum \mathbb{S} is. The reason for this phenomenon is the influence of the symmetric group actions of the spectrum. This is even more apparent in the next example:

Example 2.7. Let G be the symmetric group Σ_n and M be the natural Σ_n -set \underline{n} , X any symmetric spectrum with trivial Σ_n -action. Then $X(\underline{n})$ is canonically isomorphic to X_n but now carries the Σ_n -action coming from the data of the underlying symmetric spectrum. In contrast, evaluating at $\{1, \dots, n\}$ with *trivial* Σ_n -action yields X_n with trivial action.

These evaluations are connected by so-called generalized structure maps. Let G be a finite group, M and N two finite G -sets of orders m and n , respectively, and X a G -symmetric spectrum. We further choose a bijection $\psi : \underline{n} \xrightarrow{\cong} N$.

Definition 2.8 (Generalized structure map). The map

$$\begin{aligned} \sigma_M^N : X(M) \wedge S^N &\rightarrow X(M \sqcup N) \\ ([x \wedge f] \wedge s) &\mapsto [\sigma_m^n(x \wedge \psi_*^{-1}(s)) \wedge (f \sqcup \psi)] \end{aligned}$$

is called the *generalized structure map* of M and N .

It is straightforward to check:

Lemma 2.9. *The generalized structure map does not depend on the choice of bijection $\psi : \underline{n} \xrightarrow{\cong} N$. Furthermore, it is G -equivariant for the diagonal G -action on $X(M) \wedge S^N$.*

Here, the second statement is a consequence of σ_m^n being $\Sigma_m \times \Sigma_n$ -equivariant.

Remark 2.10. In fact we will also need the generalized structure map in the situation where G acts on $M \sqcup N$ but not necessarily in a way such that M and N are preserved by G . In this case G does not act on $X(M) \wedge S^N$, but it acts on the wedge over all $X(\alpha(M)) \wedge S^{(M \sqcup N) - \alpha(M)}$ for injections $M \rightarrow M \sqcup N$ (cf. Section 2.7). The map from this wedge to $X(M \sqcup N)$ obtained via the various generalized structure maps is G -equivariant.

2.3 G -orthogonal spectra

As we will be comparing model structures on G -symmetric spectra to ones on G -orthogonal spectra in Section 5.1, we quickly explain the analogous constructions there. G -orthogonal spectra were introduced in [MM02], other references are [Sto11] and [Sch11].

Orthogonal spectra are defined similarly to symmetric spectra of spaces with orthogonal groups $O(n)$ in place of the symmetric groups Σ_n , i.e., an orthogonal spectrum is a sequence of based $O(n)$ -spaces X_n with structure maps $X_n \wedge S^1 \rightarrow X_{n+1}$ whose iterations $X_n \wedge S^m \rightarrow X_{n+m}$ are $O(n) \times O(m)$ -equivariant.

An orthogonal spectrum X has an underlying symmetric spectrum of spaces UX by restricting the $O(n)$ -actions to Σ_n -actions via the embedding as permutation matrices. The resulting functor $U : Sp^O \rightarrow Sp_T^\Sigma$ has a left adjoint L , which can be obtained via a left Kan extension (see [MMSS01, I.3 and III.23] for a description). The left adjoint L is strong symmetric monoidal with respect to the smash products, whereas U is lax symmetric monoidal.

Definition 2.11. A G -orthogonal spectrum is a G -object in orthogonal spectra.

Remark 2.12. This is not exactly the definition of [MM02], but there is an equivalence of categories (cf. [MM02, V, Thm. 1.5], [HHR14, Prop. A.18] and Section 2.7 for the corresponding story for G -symmetric spectra).

Every G -orthogonal spectrum X can be evaluated on all finite dimensional G -representations V via the formula $X(V) = X_{\dim V} \wedge_{O(\dim V)} L(\mathbb{R}^{\dim V}, V)_+$, where L denotes the space of linear isometries, and there are G -equivariant generalized structure maps $\sigma_V^W : X(V) \wedge S^W \rightarrow X(V \oplus W)$.

These evaluations for G -symmetric and G -orthogonal spectra are related in the following way:

Lemma 2.13. *For every finite G -set M and every G -orthogonal spectrum X there is a natural G -homeomorphism $UX(M) \cong X(\mathbb{R}[M])$.*

Proof. Let m be the order of M . Then the map sending a pair $[x \in X_m, \varphi : \underline{m} \cong M]$ to $[x, \mathbb{R}[\varphi] : \mathbb{R}^m \cong \mathbb{R}[M]]$ is an equivariant homeomorphism. \square

2.4 Mandell's equivariant symmetric spectra

We also quickly recall Mandell's definition of an equivariant symmetric spectrum. As mentioned in the introduction, the underlying category depends on a choice of normal subgroup N and models the equivariant stable homotopy category with respect to the N -fixed universe.

Definition 2.14 (Mandell, introduction of [Man04]). A $G\Sigma_{G/N}$ -spectrum consists of a sequence of based $(G \times \Sigma_n)$ -simplicial sets $X(n \times (G/N))$ and structure maps $\sigma_{n \times (G/N)}^{G/N} : X(n \times (G/N)) \wedge S^{G/N} \rightarrow X((n+1) \times G/N)$ such that for all $n, m \in \mathbb{N}$ the iterate $X(n \times (G/N)) \wedge S^{m \times G/N} \rightarrow X((n+m) \times (G/N))$ is $(G \times \Sigma_n \times \Sigma_m)$ -equivariant.

The notation we used in the definition makes it clear how it relates to our version of G -symmetric spectra: Every G -symmetric spectrum of simplicial sets X gives rise to a $G\Sigma_{G/N}$ -spectrum (which we denote by $U_{G/N}X$) by only remembering the evaluations $X(n \times (G/N))$ (together with the respective generalized structure maps) and restricting the $(G \times \Sigma_{n \times (G/N)})$ -action to the diagonal $(G \times \Sigma_n)$ -action. In fact, both categories are enriched functor categories (cf. Section 2.7) and the functor $U_{G/N}$ comes from restriction along an embedding of the indexing categories. Hence, there exists a left adjoint $L_{G/N} : G\Sigma_{G/N} \rightarrow GSp_S^\Sigma$ given by left Kan extension. This left adjoint is again strong symmetric monoidal with respect to the smash products.

2.5 Functors on G -symmetric spectra

In this section we explain various functors on the category of G -symmetric spectra. The first class of examples describes lifts from functors on based G -spaces/ G -simplicial sets:

Every \mathbf{T}_* -enriched functor $F : G\mathbf{T}_* \rightarrow K\mathbf{T}_*$ for possibly different finite groups K and G can be lifted to a functor (of the same name) $F : GSp_{\mathbf{T}}^{\Sigma} \rightarrow KSp_{\mathbf{T}}^{\Sigma}$ via $F(X)_n := F(X_n)$. The structure map $F(X_n) \wedge S^1 \rightarrow F(X_{n+1})$ is adjoint to

$$S^1 \xrightarrow{\tilde{\sigma}_n} \text{map}_G(X_n, X_{n+1}) \xrightarrow{F} \text{map}_K(F(X_n), F(X_{n+1})),$$

where $\tilde{\sigma}_n$ is adjoint to the structure map of X . Moreover, if two such functors on the space-level form an adjunction, then so do their spectrum-level versions. The same applies to G -symmetric spectra of simplicial sets, with \mathbf{T}_* replaced by \mathcal{S}_* .

The following examples arise via this construction. We omit the index \mathbf{T} or \mathcal{S} , as all of them make sense in both categories.

- The trivial action functor $Sp^{\Sigma} \rightarrow GSp^{\Sigma}$ with its left adjoint G -orbits $(-)/G$ and its right adjoint G -fixed points $(-)^G$.
- The smash product $A \wedge -$ for a based G -space/ G -simplicial set A and its right adjoint $\text{map}(A, -)$.
- The restriction functor $\text{res}_H^G : GSp^{\Sigma} \rightarrow HSp^{\Sigma}$ for a subgroup $H \leq G$ with its left adjoint $G \ltimes_H -$ and its right adjoint $\text{map}_H(G, -)$.

Definition 2.15 (Geometric realization and singular complex). The geometric realization $|X|$ of a G -symmetric spectrum of simplicial sets X is a G -symmetric spectrum of spaces with n -th level $|X|_n := |X_n|$ and structure map $|X|_n \wedge S^1 \cong |X_n \wedge S^1| \xrightarrow{|\sigma_n|} |X_{n+1}| = |X|_{n+1}$. As in the unstable case, geometric realization is left adjoint to the singular complex functor $\mathcal{S} : Sp_{\mathbf{T}}^{\Sigma} \rightarrow Sp_{\mathcal{S}}^{\Sigma}$. The singular complex $\mathcal{S}X$ of a G -symmetric spectrum of spaces X is given in level n by $(\mathcal{S}X)_n := \mathcal{S}X_n$ with structure map adjoint to $(\mathcal{S}X)_n = \mathcal{S}X_n \xrightarrow{S\tilde{\sigma}_n} \mathcal{S}(\Omega X_{n+1}) \cong \Omega \mathcal{S}X_{n+1} = \Omega(\mathcal{S}X)_{n+1}$, where $\tilde{\sigma}_n$ denotes the adjoint structure map of X .

Definition 2.16 (Shift). An endofunctor of G -symmetric spectra that does not arise through a levelwise construction is the *shift* along a finite G -set M . It is defined by $(sh^M X)_n := X(M \sqcup \underline{n})$ with Σ_n acting through \underline{n} . The structure map

$$\sigma_n : (sh^M X)_n \wedge S^1 = X(M \sqcup \underline{n}) \wedge S^1 \rightarrow X(M \sqcup \underline{n+1}) = (sh^M X)_{n+1}$$

is the generalized structure map $\sigma_{M \sqcup \underline{n}}^1$ of X .

For all G -sets M there is a natural map $\alpha_X^M : S^M \wedge X \rightarrow sh^M X$ given in level n by the composite

$$S^M \wedge X_n \cong X_n \wedge S^M \xrightarrow{\sigma_n^M} X(\underline{n} \sqcup M) \xrightarrow{X(\tau_{\underline{n}, M})} X(M \sqcup \underline{n}) = (sh^M X)_n.$$

Here, the notation $X(\tau_{\underline{n}, M})$ stands for the G -map induced from the symmetry isomorphism $\tau_{\underline{n}, M}$ of the G -sets $\underline{n} \sqcup M$ and $M \sqcup \underline{n}$.

2.6 Enrichments and smash product

For two G -symmetric spectra X and Y over spaces the set of all (not necessarily equivariant) spectra-maps from X to Y carries a natural topology as the subspace of the product $\prod_n \text{map}(X_n, Y_n)$. We denote this (based at the constant map) space by $\text{map}_{Sp_{\mathbf{T}}^{\Sigma}}(X, Y)$.

The simplicial mapping set $\text{map}_{Sp_S^\Sigma}(X, Y)$ between two G -symmetric spectra of simplicial sets X and Y has n -simplices $\text{map}_{Sp_S^\Sigma}(X, Y)_n := Sp_S^\Sigma(\underline{\Delta}_+^n \wedge X, Y)$ with simplicial functoriality through $\underline{\Delta}$. By definition, the 0-simplices of $\text{map}(X, Y)$ equal the set of spectra-maps from X to Y , again based at the constant map.

Both $\text{map}_{Sp_T^\Sigma}(X, Y)$ and $\text{map}_{Sp_S^\Sigma}(X, Y)$ carry a G -action via conjugation, the G -fixed points equal the subspace (subsimplicial set) $\text{map}_{GSp_T^\Sigma}(X, Y)$ (resp. $\text{map}_{GSp_S^\Sigma}(X, Y)$) of G -equivariant spectra maps. Both notions of mapping objects give enrichments of the category of G -symmetric spectra, one over based G -spaces (respectively based G -simplicial sets) and one over based spaces (respectively based simplicial sets).

This can be extended to an enrichment over G -symmetric spectra themselves via

$$\text{Hom}(X, Y)_n = \text{map}_{Sp^\Sigma}(X, sh^n Y)$$

with Σ_n -action through sh^n and adjoint structure map

$$\text{map}_{Sp^\Sigma}(X, sh^n Y) \xrightarrow{\text{map}_{Sp^\Sigma}(X, \tilde{\alpha}_{sh^n Y}^1)} \text{map}_{Sp^\Sigma}(X, \Omega sh(sh^n Y)) \cong \Omega \text{map}_{Sp^\Sigma}(X, sh^{n+1} Y).$$

The category of symmetric spectra (both over spaces and simplicial sets) has a closed symmetric monoidal smash product (with right adjoint $\text{Hom}(-, -)$), as constructed in [HSS00, Section 2]. Hence, the category of G -symmetric spectra also inherits a symmetric monoidal product by forming the smash product of the underlying non-equivariant spectra and giving it the diagonal G -action. The following is a formal consequence:

Proposition 2.17 (Adjunction smash product/internal hom). *For all G -symmetric spectra X, Y and Z there is a (natural) G -isomorphism*

$$\text{map}_{Sp^\Sigma}(X \wedge Y, Z) \cong \text{map}_{Sp^\Sigma}(X, \text{Hom}(Y, Z)).$$

2.7 Free G -symmetric spectra and G -symmetric spectra as enriched functors

For every finite G -set M the evaluation functors $-(M) : Sp_T^\Sigma \rightarrow G\mathbf{T}_*$ and $-(M) : Sp_S^\Sigma \rightarrow G\mathcal{S}_*$ have left adjoints called *free G -symmetric spectra*, which we describe now.

Given two finite sets M and N we set

$$\Sigma(M, N) = \bigvee_{\alpha: M \rightarrow N \text{ injective}} S^{N-\alpha(M)},$$

where $N - \alpha(M)$ is the complement of the image of α in N . This comes with two group actions: a right one by Σ_M via precomposition on the indexing set of injective maps and identities on the spheres and a left one by Σ_N where an element $\sigma \in \Sigma_N$ maps a sphere $S^{N-\alpha(M)}$ associated to $\alpha : M \rightarrow N$ to the one associated to $\sigma\alpha : M \rightarrow N$ via the induced action of $\sigma|_{N-\alpha(M)} : N - \alpha(M) \rightarrow N - (\sigma\alpha)(M)$. In the case where M and N are G -sets, we can pull back this $\Sigma_N \times \Sigma_M^{op}$ -action to get a conjugation action by G on $\Sigma(M, N)$. If M and N have the same number of elements, $\Sigma(M, N)$ is canonically isomorphic to $\text{Bij}(M, N)_+$, the discrete space/simplicial set of bijections from M to N with an added basepoint. We note that the geometric realization of the G -simplicial set $\Sigma(M, N)$ is G -homeomorphic to the topological $\Sigma(M, N)$ and thus the latter possesses the structure of a G -CW complex.

For another finite set K there is a composition map

$$\circ : \Sigma(M, N) \wedge \Sigma(N, K) \rightarrow \Sigma(M, K)$$

defined via the formula $(\alpha; x) \circ (\beta; y) := (\beta\alpha; \beta(x) \wedge y)$. These composition maps are associative, unital (with respect to the identity in $\Sigma(M, M) \cong \text{Bij}(M, M)_+$) and G -equivariant. We obtain categories called $G\Sigma_{\mathbf{T}}$ and $G\Sigma_{\mathcal{S}}$, enriched over based G -spaces and based G -simplicial sets, respectively. The objects are finite G -sets and the morphisms between M and N are $\Sigma(M, N)$.

Every G -symmetric spectrum X of spaces or simplicial sets gives rise to a $G\mathbf{T}_*$ -enriched functor $G\Sigma_{\mathbf{T}} \rightarrow G\mathbf{T}_*$ (respectively $G\mathcal{S}_*$ -enriched functor $G\Sigma_{\mathcal{S}} \rightarrow G\mathcal{S}_*$) in the following way: A finite G -set M is sent to $X(M)$, the evaluation of X on M . The map on morphisms $\Sigma(M, N) \rightarrow \text{map}(X(M), X(N))$ is the adjoint of

$$\begin{aligned} X(M) \wedge \Sigma(M, N) &\longrightarrow X(N) \\ x \wedge (\alpha; y) &\mapsto X(\alpha \sqcup i)(\sigma_M^{N-\alpha(M)}(x \wedge y)). \end{aligned}$$

Here, $X(\alpha \sqcup i)$ is the isomorphism $X(M \sqcup (N - \alpha(M))) \cong X(N)$ induced from $\alpha \sqcup i : M \sqcup (N - \alpha(M)) \rightarrow N$ in the way described in Section 2.2 and we have made use of the generalized structure maps $\sigma_M^{N-\alpha(M)}$ in the sense of Remark 2.10.

In fact, it is possible to show that this construction defines an equivalence from the category of G -symmetric spectra to the category of $G\mathbf{T}_*$ -enriched functors $\text{Fun}_{G\mathbf{T}_*}(G\Sigma_{\mathcal{T}}, G\mathcal{T})$ (respectively $G\mathcal{S}_*$ -enriched functors $\text{Fun}_{G\mathcal{S}_*}(G\Sigma_{\mathcal{S}}, G\mathcal{S})$). The inverse functor is the restriction to trivial G -sets $\{1, \dots, n\}$ and the sphere associated to the inclusion $\{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$.

Definition 2.18 (Free G -symmetric spectra). Let A be a based G -space/ G -simplicial set and M a finite G -set. The *free G -symmetric spectrum on A in level M* is denoted by $\mathcal{F}_M A$ and defined via

$$(\mathcal{F}_M A)_n := A \wedge \Sigma(M, \underline{n})$$

with diagonal G -action and Σ_n -action through $\Sigma(M, \underline{n})$. The structure map is the composition

$$A \wedge \Sigma(M, \underline{n}) \wedge S^1 \hookrightarrow A \wedge \Sigma(M, \underline{n}) \wedge \Sigma(\underline{n}, \underline{n+1}) \xrightarrow{A \wedge \circ} A \wedge \Sigma(M, \underline{n+1}).$$

Here, the first map is induced by the embedding $S^1 \hookrightarrow \Sigma(\underline{n}, \underline{n+1})$ as the sphere associated to the inclusion $\underline{n} \hookrightarrow \underline{n+1}$.

We note that more generally $(\mathcal{F}_M A)(N)$ is canonically G -isomorphic to $A \wedge \Sigma(M, N)$ with diagonal G -action. When it is unclear with respect to which finite group the free spectrum is formed, we write $\mathcal{F}_M^{(G)} A$.

Example 2.19. There is exactly one injective map from the empty set to any other set. Therefore, the G -symmetric spectrum $\mathcal{F}_\emptyset A$ is naturally isomorphic to the suspension spectrum $\Sigma^\infty A$.

Proposition 2.20 (Adjunction between free spectra and evaluation). *Let M be a finite G -set, A a based G -space (or based G -simplicial set) and Y a G -symmetric spectrum. Then the natural map $\text{map}_{Sp\Sigma}(\mathcal{F}_M A, Y) \rightarrow \text{map}(A, Y(M))$ that sends a (not necessarily equivariant) morphism of symmetric spectra $f : \mathcal{F}_M A \rightarrow Y$ to the composite*

$$A \cong A \wedge \{id_M\}_+ \hookrightarrow A \wedge \Sigma(M, M) \cong (\mathcal{F}_M A)(M) \xrightarrow{f(M)} Y(M)$$

is a G -isomorphism.

Viewing G -symmetric spectra as an enriched functor category, this is a consequence of the enriched Yoneda Lemma.

As described in Section 2.2, the evaluation of a G -symmetric spectrum on a G -set M carries a Σ_M -action in addition to the G -action and the two assemble to a $G \ltimes \Sigma_M$ -action. Thus, $-(M)$ can be thought of as a functor to $(G \ltimes \Sigma_M)\mathbf{T}_*$ or $(G \ltimes \Sigma_M)\mathcal{S}_*$. This functor also has a left adjoint, which in the case of G -orthogonal spectra first appeared in Stolz's thesis [Sto11, Definition 2.2.14].

Definition 2.21 (Semi-free G -symmetric spectra). Let M be a finite G -set and A a based $(G \ltimes \Sigma_M)$ -space/ $(G \ltimes \Sigma_M)$ -simplicial set. We recall that the natural Σ_M -action turns M into a $(G \ltimes \Sigma_M)$ -set. Then the *semi-free G -symmetric spectrum on A* , denoted $\mathcal{G}_M A$, is defined as the quotient of the $(G \ltimes \Sigma_M)$ -spectrum $\mathcal{F}_M^{(G \ltimes \Sigma_M)} A$ by the action of the normal subgroup Σ_M .

As advertised we have:

Proposition 2.22 (Adjunction between semi-free spectra and evaluation). *For A a based $(G \ltimes \Sigma_M)$ -space/ $(G \ltimes \Sigma_M)$ -simplicial set and Y a G -symmetric spectrum there is a natural isomorphism $\text{map}_{GSp^\Sigma}(\mathcal{G}_M A, Y) \cong \text{map}_{G \ltimes \Sigma_M}(A, Y(M))$.*

Proof. The adjunction isomorphism is given by the composition

$$\text{map}_{GSp^\Sigma}(\mathcal{G}_M A, Y) \cong \text{map}_{(G \ltimes \Sigma_M)Sp^\Sigma}(\mathcal{F}_M^{(G \ltimes \Sigma_M)} A, Y) \cong \text{map}_{G \ltimes \Sigma_M}(A, Y(M)). \quad \square$$

There is a more concrete description of $\mathcal{G}_M(A)$ in levels of the form $M \sqcup N$, where it is given by $(G \ltimes \Sigma_{M \sqcup N}) \ltimes_{G \ltimes (\Sigma_M \times \Sigma_N)} (A \wedge S^N)$. In addition, all evaluations at G -sets of lesser order than that of M are given by a point. In Section 7 we will use repeatedly that more generally the smash product of a G -symmetric spectrum X with a semi-free G -symmetric spectrum $\mathcal{G}_m(A)$ for a based $(G \ltimes \Sigma_m)$ -space/ $(G \ltimes \Sigma_m)$ -simplicial set A can be described explicitly in the following way:

$$(\mathcal{G}_m(A) \wedge X)_k \cong \begin{cases} * & \text{for } k < m \\ \Sigma_{m+n} \ltimes_{\Sigma_m \times \Sigma_n} (A \wedge X_n) & \text{for } k = m + n \end{cases}$$

This can be checked via the universal properties of the smash product and semi-free G -symmetric spectra.

2.8 Latching spaces and the skeleton filtration of a map

Every map of G -symmetric spectra $f : X \rightarrow Y$ can be factored as a countable composite

$$X = F^{-1}[f] \xrightarrow{j_0[f]} F^0[f] \xrightarrow{j_1[f]} F^1[f] \xrightarrow{j_2[f]} \dots \rightarrow Y,$$

which builds Y out of X “one dimension at a time” in a sense described below. This factorization is important for stating the model structures in Section 2.9 and for proving the lifting property axioms in them.

First we treat the absolute case and define the following for every G -symmetric spectrum X :

- For every $n \geq 0$ a based $G \ltimes \Sigma_n$ -space/ $G \ltimes \Sigma_n$ -simplicial set $L_n X$, called the *n -th latching object of X* , together with a natural *latching map* $\nu_n(X) : L_n X \rightarrow X_n$.
- For every $n \geq -1$ a G -symmetric spectrum $F^n X$, the *n -skeleton* of X , together with a natural map $i_n(X) : F^n X \rightarrow X$.
- A natural map $j_n(X) : F^{n-1} X \rightarrow F^n X$ satisfying $i_n(X) \circ j_n(X) = i_{n-1}(X)$.

For this we set $F^{-1} X = *$ and proceed inductively. Let $n \in \mathbb{N}$ and assume all the data has been defined for $k < n$. Then we define $L_n X$ to be $(F^{n-1} X)_n$, the n -th level of the $n-1$ skeleton, and the n -th latching map $\nu_n(X) : L_n X \rightarrow X_n$ to be the n -th level of $i_{n-1}(X)$. Furthermore, we define $F^n X$ as the pushout of the counit $\mathcal{G}_n(L_n X) \rightarrow F^{n-1} X$ along $\mathcal{G}_n(\nu_n(X)) : \mathcal{G}_n(L_n X) \rightarrow \mathcal{G}_n(X_n)$. The map $j_n(X)$ is taken to be the induced map into the pushout and $i_n(X)$ is induced by the universal property with respect to the counit for X and $i_{n-1}(X)$.

More carefully, since all G -symmetric spectra of the form $\mathcal{G}_n(-)$ are equal to a point in levels below n and the counit is an isomorphism in degree n , we can set $(F^n X)_k = X_k$ for all $k \leq n$.

It follows that the sequence of the $F^n X$ converges to X in the strong sense that in a fixed level all maps are eventually identities.

From this we obtain a relative version for a map $f : X \rightarrow Y$ by setting $F^n[f] = X \cup_{F^n X} F^n Y$, $L_n[f] = X_n \cup_{L_n X} L_n Y$ and $j_n[f] : F^n[f] \rightarrow F^{n+1}[f]$ as well as $i_n[f] : F^n[f] \rightarrow Y$ the induced maps from the respective absolute versions for X and Y . It follows that $F^{-1}[f] = X$, that the colimit along the $F^n[f]$ converges to Y in the same strong sense as above and that the infinite composite equals f . Furthermore, the diagram

$$\begin{array}{ccc} \mathcal{G}_n(L_n[f]) & \xrightarrow{\varepsilon_{F^{n-1}[f]}} & F^{n-1}[f] \\ \mathcal{G}_n(\nu_n[f]) \downarrow & & \downarrow j_n[f] \\ \mathcal{G}_n(Y_n) & \xrightarrow{\varepsilon_{F^n[f]}} & F^n[f] \end{array}$$

is a pushout for all n .

Remark 2.23. One should note that despite the name “filtration” it is not necessarily true that all maps $j_n(X)$ (or $i_n(X)$) appearing in the skeleton filtration are levelwise injective. An example is given by the truncated sphere spectrum (cf. [HSS00, Def. 5.2.1]).

The connection to model structures comes from the following lemma:

Lemma 2.24. *Let $f : X \rightarrow Y$ and $g : W \rightarrow Z$ be maps of G -symmetric spectra such that for all $n \geq 0$ the $(G \times \Sigma_n)$ -map $\nu_n[f] : L_n[f] \rightarrow Y_n$ has the left lifting property with respect to $g_n : W_n \rightarrow Z_n$. Then f has the left lifting property with respect to g .*

Proof. Using the adjunction between semi-free spectra and evaluation, the assumption is equivalent to $\mathcal{G}_n(\nu_n[f])$ having the left lifting property with respect to g for all n . But by the properties of the skeleton filtration above, f can be obtained from these by pushouts and countable composition, which preserve the left lifting property. \square

2.9 Level model structures

In this section we construct various level model structures on the categories of G -symmetric spectra of spaces and simplicial sets. Precisely, for every G -set universe \mathcal{U} we describe a projective and a flat (or \mathbb{S} -) model structure, in positive and non-positive versions. As mentioned in the introduction, the former is a variant of the level model structure of G -orthogonal spectra of [MM02] and the level model structure of [Man04], and the latter is a generalization of the non-equivariant flat model structure by Shipley [Shi04] and a translation from the one on G -orthogonal spectra by Stolz [Sto11].

From now on we fix a G -set universe \mathcal{U} , i.e., a countably infinite G -set which is isomorphic to the disjoint union of two copies of itself. Up to non-canonical isomorphism, such a G -set universe is always of the form $\bigsqcup_{H \in \mathcal{C}} (\mathbb{N} \times G/H)$ for a non-empty set of subgroups \mathcal{C} which becomes unique if one requires it to be closed under conjugation.

We recall from Example 1.17 that $\mathcal{F}^{G, \Sigma_n}$ is the family of subgroups of $G \times \Sigma_n$ that intersect $1 \times \Sigma_n$ only in the neutral element. Elements of this family correspond to pairs consisting of a subgroup H of G and a group homomorphism $\alpha : H \rightarrow \Sigma_n$. We denote by $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ the subfamily of $\mathcal{F}^{G, \Sigma_n}$ corresponding to those pairs (H, α) such that \underline{n} equipped with the H -set structure through α , which we denote by \underline{n}_{α} , allows an H -embedding into \mathcal{U} . The level model structures on G -symmetric spectra are constructed out of the projective and mixed model structures associated to these families for varying n (cf. Section 1.3). We start with the non-positive versions, the positive modifications are explained in the paragraph preceding Proposition 2.37.

Definition 2.25. A map $f : X \rightarrow Y$ of G -symmetric spectra of spaces or simplicial sets is called a

- $G^{\mathcal{U}}$ -level equivalence if for all $n \in \mathbb{N}$ the $(G \times \Sigma_n)$ -map $f_n : X_n \rightarrow Y_n$ is an $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ -equivalence.
- $G^{\mathcal{U}}$ -projective (resp. $G^{\mathcal{U}}$ -flat) level fibration if for all $n \in \mathbb{N}$ the $(G \times \Sigma_n)$ -map $f_n : X_n \rightarrow Y_n$ is a projective (resp. mixed) $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ -fibration.
- $G^{\mathcal{U}}$ -projective (resp. G -flat) cofibration if for all $n \in \mathbb{N}$ the pushout product map

$$\nu_n[f] : X_n \cup_{L_n X} L_n Y \rightarrow Y_n$$

is a projective $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ -cofibration (resp. genuine G -cofibration).

The class of $G^{\mathcal{U}}$ -level equivalences (and that of projective $G^{\mathcal{U}}$ -level fibrations) has a different description that motivates its definition:

Proposition 2.26. *Let $f : X \rightarrow Y$ be a map of G -symmetric spectra. Then the following are equivalent:*

1. *The map f is a $G^{\mathcal{U}}$ -level equivalence (resp. $G^{\mathcal{U}}$ -projective level fibration).*
2. *For all subgroups H of G and all finite H -subsets M of \mathcal{U} the H -map $f(M) : X(M) \rightarrow Y(M)$ induces a weak equivalence (resp. Serre/Kan fibration) on H -fixed points.*

By applying it for all subgroups of H with the restricted action on M , one can also replace the second condition by requiring $f(M)$ to be a genuine H -equivalence (resp. genuine H -fibration) for all finite H -subsets $M \subseteq \mathcal{U}$.

The proposition is a consequence of the following two lemmas which together show that subgroups L of $G \times \Sigma_n$ which lie in the family $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ essentially correspond to pairs of a subgroup H of G and a finite H -subset $M \subseteq \mathcal{U}$ of cardinality n and that for a G -symmetric spectrum X the respective fixed points X_n^L and $X(M)^H$ are naturally isomorphic. The names “Twisting” and “Untwisting” are taken from [Sto11, Section 2.3.3].

Lemma 2.27 (Untwisting). *Let n be a natural number and K a subgroup of $G \times \Sigma_n$ that lies in the family $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$. Then there exists a subgroup H of G , a group isomorphism $j : H \rightarrow K$ and an H -set structure on \underline{n} which embeds into \mathcal{U} , such that there is an H -isomorphism*

$$j^*(X_n) \cong X(\underline{n})$$

for every G -symmetric spectrum X . This H -isomorphism can be chosen natural in X .

Proof. We already know the existence of a subgroup H of G and a group homomorphism $\varphi : H \rightarrow \Sigma_n$, such that K is equal to $\{(h, \varphi(h)) \mid h \in H\}$. Thus, the map j that sends $h \in H$ to the element $(h, \varphi(h)) \in K$ is an isomorphism. In addition, φ corresponds to an H -set structure on \underline{n} and we have seen in Section 2.2 on evaluation that there is a natural H -isomorphism $X(\underline{n}) \cong X_n$ where the action on the latter is precisely given by restriction along j . \square

Lemma 2.28 (Twisting). *Let H be a subgroup of G and M an H -set of order n which embeds into \mathcal{U} . Then there exists a subgroup K of $G \times \Sigma_n$ in the family $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ and an isomorphism $j : H \rightarrow K$, such that there is an H -isomorphism $X(M) \cong j^*(X_n)$ for every G -symmetric spectrum X . This H -isomorphism can be chosen natural in X .*

Proof. We begin by choosing a bijection from M to \underline{n} . As we have seen in Section 2.2, this induces an H -set structure on \underline{n} and a natural H -isomorphism $X(M) \cong X(\underline{n})$. Furthermore, we know that, letting $\varphi : H \rightarrow \Sigma_n$ be the corresponding homomorphism, there is another natural H -isomorphism $X(\underline{n}) \cong X_n$, where the action on the latter is restriction along the homomorphism $j := (i, \varphi) : H \rightarrow G \times \Sigma_n$. This homomorphism is injective and its image K lies in the family $\mathcal{F}_U^{G, \Sigma_n}$, as desired. \square

In another common definition of level equivalence of G -spectra, for example in the translation of the model structure in [MM02] to the symmetric context, one only requires $f(M)$ to be an equivalence on H -fixed points for H -sets M that are restrictions of G -sets. This is more natural if one views G -symmetric spectra as enriched functor categories as explained in Section 2.7. The stronger notion we use has the advantage that for a subgroup H of G the restriction functor from G -symmetric spectra to H -symmetric spectra preserves fibrations and weak equivalences and thus becomes a right Quillen functor. For G -orthogonal spectra, this stronger version is also used in [Sto11, Def. 2.3.5] and the positive one in [HHR14, B.4.].

We now proceed to proving the model category axioms. By definition, the classes of $G^{\mathcal{U}}$ -projective/ $G^{\mathcal{U}}$ -flat cofibrations, fibrations and $G^{\mathcal{U}}$ -level equivalences are formed out of the respective classes for the projective/mixed $\mathcal{F}_U^{G, \Sigma_n}$ -model structures in each dimension. We use a more general proposition that is - together with its proof - close to [Sch13, Section III.2 on level model structures]. The reader will notice that few specific properties of based G -spaces or based G -simplicial sets are used and that the construction would in fact work for symmetric spectra over any pointed simplicially enriched, tensored and cotensored category.

We first define:

Definition 2.29 (Consistency condition). For every $n \in \mathbb{N}$ let \mathcal{M}_n be a model structure on based $(G \times \Sigma_n)$ -spaces/ $(G \times \Sigma_n)$ -simplicial sets. The collection of model structures $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ is said to satisfy the *consistency condition* if for all $m, n \in \mathbb{N}$ and every acyclic cofibration $f : A \rightarrow B$ in \mathcal{M}_m the pushout of the $(G \times \Sigma_n)$ -map $f \wedge_{\Sigma_m} \Sigma(\underline{m}, \underline{n})$ along an arbitrary $(G \times \Sigma_n)$ -map $g : A \wedge_{\Sigma_m} \Sigma(\underline{m}, \underline{n}) \rightarrow X$ is a weak equivalence in \mathcal{M}_n .

In particular, the consistency condition holds if the functors

$$- \wedge_{\Sigma_m} \Sigma(\underline{m}, \underline{n}) : (G \times \Sigma_m) \mathbf{T}_* \rightarrow (G \times \Sigma_n) \mathbf{T}_*$$

(or their simplicial analogs) are left Quillen functors, in which case we say that the \mathcal{M}_n satisfy the *strong consistency condition*.

The following proposition says that given the consistency condition is satisfied, the model structures \mathcal{M}_n assemble to a level model structure on G -symmetric spectra.

Proposition 2.30 (Level model structures). *Let $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ be a collection of model structures on based $(G \times \Sigma_n)$ -spaces/ $(G \times \Sigma_n)$ -simplicial sets satisfying the consistency condition. We call a map $f : X \rightarrow Y$ of G -symmetric spectra a level weak equivalence if for all $n \in \mathbb{N}$ the map f_n is an \mathcal{M}_n -weak-equivalence, a level fibration if for all $n \in \mathbb{N}$ the map f_n is an \mathcal{M}_n -fibration and a level cofibration if for all $n \in \mathbb{N}$ the pushout product $\nu_n[f] : X_n \cup_{L_n X} L_n Y \rightarrow Y_n$ is an \mathcal{M}_n -cofibration.*

Then these classes define a model structure on G -symmetric spectra. If all model structures \mathcal{M}_n are right proper, then so is this model structure. If all model structures \mathcal{M}_n are left proper and satisfy the strong consistency condition, then this model structure is also left proper.

The purpose of the consistency condition lies in the following:

Lemma 2.31. *Let $f : X \rightarrow Y$ be a map of G -symmetric spectra such that all latching maps $\nu_n[f]$ are acyclic cofibrations in the respective model structures \mathcal{M}_n . Then f is an acyclic cofibration of G -symmetric spectra.*

Proof. We only have to show that f is a weak equivalence since it is a cofibration by definition. As we saw in Section 2.8, each component f_n is a finite composition of pushouts of

$$(\mathcal{G}_m(\nu_m[f]))_n = \nu_m[f] \wedge_{\Sigma_m} \Sigma(\underline{m}, \underline{n})$$

for $m \leq n$. These maps are by the assumption on f and the consistency condition weak equivalences, hence so is f_n . \square

Proof of Proposition 2.30. As was explained in Section 2.1, G -symmetric spectra have all limits and colimits. It follows from the respective properties in the model structures \mathcal{M}_n that weak equivalences possess the 2-out-of-3 property and that weak equivalences and fibrations are closed under retracts. The n -th latching map is a functor on the arrow category and thus turns retract diagrams into retract diagrams and so the retract of a cofibration is also one.

The factorizations are constructed step by step using those for the model categories \mathcal{M}_n . Let $f : X \rightarrow Y$ be a map of G -symmetric spectra. We start by factoring f_0 as a cofibration $i_0 : X_0 \rightarrow A_0$ followed by an acyclic fibration $p_0 : A_0 \rightarrow Y_0$ in \mathcal{M}_0 . Now we let $n \geq 1$ and assume we have inductively defined based $G \times \Sigma_m$ -spaces/ $G \times \Sigma_m$ -simplicial sets A_m for $m < n$ together with structure maps $A_{m-1} \wedge S^1 \rightarrow A_m$ and factorizations $X_m \xrightarrow{i_m} A_m \xrightarrow{p_m} Y_m$. From this data one can already construct the n -th latching object $L_n A$ and the map $L_n A \rightarrow Y_n$ that assembles with $f_n : X_n \rightarrow Y_n$ to give a map $X_n \cup_{L_n X} L_n A \rightarrow Y_n$. We factor this map as

$$X_n \cup_{L_n X} L_n A \xrightarrow{j_n} A_n \xrightarrow{p_n} Y_n$$

where j_n is a cofibration and p_n an acyclic fibration in the model structure \mathcal{M}_n . We then set i_n as the restriction of j_n to X_n and let the structure map $A_{n-1} \wedge S^1 \rightarrow A_n$ be the composition of $A_{n-1} \wedge S^1 \rightarrow L_n A$ with the restriction of j_n to $L_n A$.

After finishing this process we have obtained a factorization $X \xrightarrow{i} A \xrightarrow{p} Y$ where p is by construction an acyclic fibration. The n -th latching morphism of i equals the map j_n from the construction, which is chosen as a cofibration in \mathcal{M}_n . Hence, i is a level cofibration of G -symmetric spectra.

We can construct another factorization $X \xrightarrow{i'} A \xrightarrow{p'} Y$ through this procedure by factoring the map $X_n \cup_{L_n X} L_n A \rightarrow Y_n$ in the induction process by an acyclic cofibration followed by a fibration. So p' is a fibration and i' has the property that all latching maps are acyclic cofibrations. By Lemma 2.31, it is an acyclic cofibration and we have established both factorizations.

It is a consequence of Lemma 2.24 and the lifting properties in the model structures \mathcal{M}_n that every cofibration has the left lifting property with respect to every acyclic fibration. The same lemma also implies that every map for which all latching maps are acyclic cofibrations has the left lifting property with respect to fibrations. We now show that every acyclic cofibration has this property (and thus together with Lemma 2.31 these two classes agree). Let $f : X \rightarrow Y$ be an acyclic cofibration of G -symmetric spectra. As shown in the last paragraph, one can factor $f = p'i'$ with p' a fibration and i' with the property that all latching maps are acyclic cofibrations. We also know (cf. Lemma 2.31) that i' is a weak equivalence, thus p' is in fact an acyclic fibration. Since f is in particular a cofibration, the commutative square

$$\begin{array}{ccc} X & \xrightarrow{i'} & A \\ f \downarrow & \nearrow & \downarrow p' \\ Y & \xrightarrow{id_Y} & Y \end{array}$$

possesses a lift that establishes f as a retract of i' . Thus, every latching map of f is also a retract of that of i' and therefore an acyclic cofibration. Therefore, f has the left lifting property with respect to all fibrations and we have proved that all model category axioms are satisfied.

Finally, the statement about right properness follows directly from the definition, since weak equivalences and fibrations are defined levelwise and the pullback is computed levelwise. The same argument works for left properness, but one first has to note that all components f_n of a level cofibration f are cofibrations in the respective model structure \mathcal{M}_n . This is a consequence of the strong consistency condition, since it guarantees that $(-)\wedge_{\Sigma_m}\Sigma(\underline{m}, \underline{n})$ preserves cofibrations. The n -th level of a cofibration f is the finite composition of pushouts of $\nu_m[f] \wedge_{\Sigma_m}\Sigma(\underline{m}, \underline{n})$ for $m \leq n$ (cf. Section 2.8) and hence a cofibration in the model structure \mathcal{M}_n . Therefore, the resulting model structure on G -symmetric spectra is left proper, too, and the proposition is proven. \square

In order to show that the classes of $G^{\mathcal{U}}$ -level equivalences, $G^{\mathcal{U}}$ -projective/ $G^{\mathcal{U}}$ -flat level fibrations and $G^{\mathcal{U}}$ -projective/ G -flat cofibrations as defined at the beginning of the section assemble to a model structure on G -symmetric spectra, it now remains to check that they satisfy the consistency condition. Since the two share the same weak equivalences, it suffices to prove:

Proposition 2.32. *The mixed model structures on $(G \times \Sigma_n)$ -spaces/ $(G \times \Sigma_n)$ -simplicial sets with respect to the families $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ satisfy the strong consistency condition.*

Proof. For $m > n$ the required condition holds trivially since in that case $\Sigma(\underline{m}, \underline{n}) = *$. Otherwise, we write $n = m + k$ with $k \geq 0$. The map

$$\begin{aligned} \Sigma_{m+k} \ltimes_{\Sigma_k} S^k &\longrightarrow \Sigma(m, m+k) \\ (\sigma, x) &\mapsto (\sigma|_m; (\sigma|_k)_*(x)) \end{aligned}$$

is a $\Sigma_{m+k} \times \Sigma_m^{op}$ -equivariant isomorphism. Thus, the functor $- \wedge_{\Sigma_m} \Sigma(m, m+k)$ is naturally isomorphic to $\Sigma_{m+k} \ltimes_{\Sigma_m \times \Sigma_k} (- \wedge S^k)$. Hence, it sends genuine $(G \times \Sigma_m)$ -cofibrations to genuine $(G \times \Sigma_{m+k})$ -cofibrations.

For acyclic cofibrations we first treat the case of spaces. It suffices to check the property on the set of generators $J_{G, mix}^{\mathcal{F}_{\mathcal{U}}^{G, \Sigma_m}}$ introduced in Section 1.3. Every map in this set is a $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_m}$ -equivalence between $(G \times \Sigma_m)$ -CW complexes. Hence the statement follows by putting $A = S^k$ in Lemma A.1.

Finally, since a map of $(G \times \Sigma_m)$ -simplicial sets is an $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_m}$ -equivalence and genuine cofibration if and only if its geometric realization is, the simplicial case follows from the topological one. \square

Hence, we obtain:

Corollary 2.33 (Flat level model structure on G -symmetric spectra). *The classes of G -flat cofibrations, $G^{\mathcal{U}}$ -flat level fibrations and $G^{\mathcal{U}}$ -level equivalences define a proper model structure on the category of G -symmetric spectra of spaces or simplicial sets.*

As well as:

Corollary 2.34 (Projective level model structure on G -symmetric spectra). *The classes of $G^{\mathcal{U}}$ -projective cofibrations, $G^{\mathcal{U}}$ -projective level fibrations and $G^{\mathcal{U}}$ -level equivalences define a proper model structure on the category of G -symmetric spectra of spaces or simplicial sets.*

In the projective case the left properness does not directly follow from Proposition 2.30, but it is a consequence of the left properness of the flat level model structure, since every $G^{\mathcal{U}}$ -projective cofibration is also a G -flat cofibration and the weak equivalences are the same.

Remark 2.35. We note that both the flat and the projective \mathcal{U} -level model structure only depend on the isomorphism type of \mathcal{U} and not \mathcal{U} itself.

By the adjunction between (semi-)free spectra and evaluation the flat and the projective \mathcal{U} -level model structures on G -symmetric spectra are cofibrantly generated with

$$I_{\mathcal{U},fl}^{lev} := \{\mathcal{G}_n(i) \mid n \in \mathbb{N}, i \in I_{G \times \Sigma_n}\} \quad (2.1)$$

and

$$J_{\mathcal{U},fl}^{lev} := \{\mathcal{G}_n(j) \mid n \in \mathbb{N}, j \in J_{G \times \Sigma_n, mix}^{\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}}\}, \quad (2.2)$$

respectively

$$I_{\mathcal{U},proj}^{lev} := \{G \ltimes_H (\mathcal{F}_M^{(H)}(i)) \mid H \leq G, M \subseteq_H \mathcal{U}, i \in I_H\} \quad (2.3)$$

and

$$J_{\mathcal{U},proj}^{lev} := \{G \ltimes_H (\mathcal{F}_M^{(H)}(j)) \mid H \leq G, M \subseteq_H \mathcal{U}, j \in J_H\} \quad (2.4)$$

as generating cofibrations and acyclic cofibrations.

Remark 2.36. It follows by adjunction that all free G -symmetric spectra $\mathcal{F}_M A$ are G -flat, provided that A is a G -cofibrant in the topological case. If M is contained in \mathcal{U} they are also $G^{\mathcal{U}}$ -projective. Similarly, the semi-free G -spectra $\mathcal{G}_M A$ for cofibrant $(G \ltimes \Sigma_M)$ -spaces A are G -flat. Whether they are $G^{\mathcal{U}}$ -projective depends on the isotropy of A and \mathcal{U} .

As in the non-equivariant case, in order to obtain model structures on the category of commutative G -symmetric ring spectra, we need a positive variant of these level model structures. It is obtained via Proposition 2.30 by replacing the genuine model structure \mathcal{M}_0 on based $G \times \Sigma_0 = G$ -spaces/ G -simplicial sets in level 0 by the one where the cofibrations are the isomorphisms and the weak equivalences and fibrations are arbitrary maps. Concretely, this means that the positive cofibrations form a subclass of the cofibrations in the respective non-positive version, consisting of those where the 0-th latching map is an isomorphism. The weak equivalences and the fibrations need to satisfy the same conditions as their non-positive analogs, except for the one in degree 0. It is immediate that these also satisfy the consistency condition and so we get:

Proposition 2.37 (Positive level model structures). *The classes of positive G -flat (positive $G^{\mathcal{U}}$ -projective) cofibrations, positive $G^{\mathcal{U}}$ -flat (resp. positive $G^{\mathcal{U}}$ -projective) fibrations and positive $G^{\mathcal{U}}$ -level equivalences form a proper cofibrantly generated model structure on the category of G -symmetric spectra.*

One obtains generating (acyclic) cofibrations by leaving out the maps of the form $\mathcal{G}_0(-)$ and $\mathcal{F}_0(-)$ in the generating (acyclic) cofibrations of their non-positive analogs described above.

We further have the following, where \square denotes the pushout product with respect to the smash product of G -symmetric spectra:

Proposition 2.38. *Let $i : A \rightarrow B$ and $j : C \rightarrow D$ be two maps of G -symmetric spectra. Then the following hold:*

- *If i and j are G -flat ($G^{\mathcal{U}}$ -projective) cofibrations, then so is the pushout product $i \square j$. If either i or j is positive, then the pushout product is also positive.*
- *If i is a G -flat cofibration and $G^{\mathcal{U}}$ -level equivalence and j is a G -flat cofibration, then $i \square j$ is also a $G^{\mathcal{U}}$ -level equivalence.*
- *If i is a positive G -flat cofibration and positive $G^{\mathcal{U}}$ -level equivalence and j a G -flat cofibration, then $i \square j$ is a $G^{\mathcal{U}}$ -level equivalence.*

Proof. It suffices to show each of these on generating (acyclic) cofibrations. The first one follows from the natural isomorphisms $\mathcal{F}_M(f) \square \mathcal{F}_N(g) \cong \mathcal{F}_{M \sqcup N}(f \square g)$ and $\mathcal{G}_n(f) \square \mathcal{G}_m(g) \cong \mathcal{G}_{m+n}(\Sigma_{n+m} \ltimes_{\Sigma_n \times \Sigma_m} (f \square g))$ and the fact that the respective structures on equivariant spaces are closed symmetric monoidal. So do the other two in the projective case. In the flat case we need Lemma A.1 in addition, which guarantees that $\Sigma_{n+m} \ltimes_{\Sigma_n \times \Sigma_m} (f \wedge X)$ is an $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_{m+n}}$ -equivalence if X is a cofibrant $(G \times \Sigma_m)$ -space. This implies that the pushout product $\Sigma_{n+m} \ltimes_{\Sigma_n \times \Sigma_m} (f \square g)$ is also one by 2-out-of-3. \square

Corollary 2.39. *The non-positive $G^{\mathcal{U}}$ -flat and $G^{\mathcal{U}}$ -projective level model structures are monoidal with respect to the smash product of G -symmetric spectra.*

The positive ones are not quite monoidal because the unit \mathbb{S} is not positively cofibrant and in general the map $\mathbb{S}^+ \wedge X \rightarrow X$ is not a positive $G^{\mathcal{U}}$ -level equivalence, even if X is positive cofibrant. This problem will disappear in the stable model structures.

Since the suspension spectrum functors $G\mathbf{T}_* \rightarrow GSp_T^{\Sigma}$ and $G\mathcal{S}_* \rightarrow GSp_S^{\Sigma}$ are left Quillen with respect to both the non-positive projective and flat model structures, Proposition 2.38 implies that the positive and non-positive, flat and projective \mathcal{U} -model structures are in particular based G -topological (respectively based G -simplicial).

2.10 $G^{\mathcal{U}}\Omega$ -spectra and $G^{\mathcal{U}}$ -stable equivalences

We now move towards the stable model structure, beginning with the notion of an equivariant Ω -spectrum:

Definition 2.40 ($G^{\mathcal{U}}\Omega$ -spectra). A $G^{\mathcal{U}}$ -projective level fibrant G -symmetric spectrum X is called a $G^{\mathcal{U}}\Omega$ -spectrum if for all subgroups H of G and all finite H -subsets M and N of \mathcal{U} the adjoint generalized structure map induces a weak equivalence

$$(\tilde{\sigma}_M^N)^H : X(M)^H \rightarrow \text{map}_H(S^N, X(M \sqcup N))$$

on H -fixed points. We say that a positively $G^{\mathcal{U}}$ -projective level fibrant spectrum is a positive $G^{\mathcal{U}}\Omega$ -spectrum if it satisfies the above condition except for the cases where $M = \emptyset$.

Again we remark that there is another common notion of Ω -spectrum for G -spectra in which one only requires the map $\tilde{\sigma}_{M,N}^H : X(M)^H \rightarrow \text{map}_H(S^N, X(M \sqcup N))$ to be a weak equivalence for H -sets M and N that are restrictions of G -sets. With this other definition the restriction of a $G^{\mathcal{U}}\Omega$ -spectrum to a subgroup H is not necessarily an $H^{\mathcal{U}}\Omega$ -spectrum, which is the case for our notion. This is related to the two notions of $G^{\mathcal{U}}$ -level equivalences mentioned in the last section.

Example 2.41. Let N be a finite G -set contained in \mathcal{U} and A a cofibrant G -space. Then the endofunctors sh^N and $\text{map}(A, -)$ preserve (positive) $G^{\mathcal{U}}\Omega$ -spectra.

Corollary 2.42. *If X is a $G^{\mathcal{U}}\Omega$ -spectrum, then so is $\Omega^N sh^N X$ for any finite G -set N contained in \mathcal{U} and the natural map $\tilde{\alpha}_X^N : X \rightarrow \Omega^N sh^N X$ is a $G^{\mathcal{U}}$ -level equivalence.*

Proof. Evaluated on a finite H -set $M \subseteq \mathcal{U}$ for some subgroup H of G , the map $\tilde{\alpha}_X^N$ is given by

$$X(M) \xrightarrow{\tilde{\sigma}_M^N} \Omega^N X(M \sqcup N) \xrightarrow{\Omega^N X(\tau_{M,N})} \Omega^N X(N \sqcup M) = (\Omega^N sh^N X)(M)$$

and thus the composition of a genuine H -equivalence and an H -isomorphism. \square

Corollary 2.43. *For every $G^{\mathcal{U}}$ -projective G -symmetric spectrum X the functor $\text{Hom}(X, -)$ preserves $G^{\mathcal{U}}\Omega$ -spectra. If X is G -flat, it preserves $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectra.*

We can now give the definition of the class of stable equivalences. We denote by $\mathrm{Ho}_{lev}^{\mathcal{U}}$ the localization of GSp^{Σ} at the class of $G^{\mathcal{U}}$ -level equivalences, and by $\gamma : GSp^{\Sigma} \rightarrow \mathrm{Ho}_{lev}^{\mathcal{U}}$ the projection functor, which can be chosen to be the identity on objects.

Definition 2.44 ($G^{\mathcal{U}}$ -stable equivalence). A map $f : X \rightarrow Y$ of G -symmetric spectra is a $G^{\mathcal{U}}$ -stable equivalence if for all $G^{\mathcal{U}}\Omega$ -spectra Z the map

$$\mathrm{Ho}_{lev}^{\mathcal{U}}(\gamma(f), Z) : \mathrm{Ho}_{lev}^{\mathcal{U}}(Y, Z) \rightarrow \mathrm{Ho}_{lev}^{\mathcal{U}}(X, Z)$$

is a bijection.

Making use of the level model structures we constructed, one can also characterize $G^{\mathcal{U}}$ -stable equivalences in the following way: Two maps of G -symmetric spectra $f, g : X \rightarrow Y$ are called G -homotopic if they lie in the same path-component of the mapping space $\mathrm{map}_{GSp^{\Sigma}}(X, Y)$. The set of G -homotopy classes of G -maps is denoted $[X, Y]^G$. Then a map $f : X \rightarrow Y$ is a $G^{\mathcal{U}}$ -stable equivalence if and only if for every $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectrum Z and some (and hence any) G -flat replacement $f^b : X^b \rightarrow Y^b$ of f the induced map

$$[f^b, Z]^G : [Y^b, Z]^G \rightarrow [X^b, Z]^G$$

is a bijection. A similar characterization can be obtained via the $G^{\mathcal{U}}$ -projective level model structure.

Remark 2.45. Since $G^{\mathcal{U}}$ -level equivalences are mapped to isomorphisms in the homotopy category, they are in particular $G^{\mathcal{U}}$ -stable equivalences. On the other hand, the Yoneda lemma implies that every $G^{\mathcal{U}}$ -stable equivalence between $G^{\mathcal{U}}\Omega$ -spectra is already a $G^{\mathcal{U}}$ -level equivalence.

Example 2.46. For a subgroup H of G and two finite H -subsets M and N of \mathcal{U} we define

$$\lambda_{M,N}^{(H)} : \mathcal{F}_{M \sqcup N}^{(H)} S^N \rightarrow \mathcal{F}_M^{(H)} S^0$$

to be the map of H -spectra adjoint to the embedding $S^N \hookrightarrow \Sigma(M, M \sqcup N) = (\mathcal{F}_M^{(H)} S^0)(M \sqcup N)$ associated to the inclusion $M \hookrightarrow M \sqcup N$.

Then the induction $G \ltimes_H \lambda_{M,N}^{(H)}$ to G -symmetric spectra is a $G^{\mathcal{U}}$ -stable equivalence.

Proof. Let Z be a $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectrum. The H -spectra $\mathcal{F}_{M \sqcup N}^{(H)} S^N$ and $\mathcal{F}_M^{(H)} S^0$ are H -flat (cf. Remark 2.36), thus their inductions to G -spectra are G -flat. So we do not need to replace them. We have the following chain of adjunction isomorphisms:

$$\begin{array}{ccccc} [G \ltimes_H \mathcal{F}_M^{(H)} S^0, Z]^G & \xrightarrow{\cong} & [\mathcal{F}_M^{(H)} S^0, Z]^H & \xrightarrow{\cong} & \pi_0(Z(M)^H) \\ \downarrow [G \ltimes_H \lambda_{M,N}^{(H)}, Z]^G & & \downarrow [\lambda_{M,N}^{(H)}, Z]^H & & \downarrow \pi_0((\bar{\sigma}_M^N)^H) \\ [G \ltimes_H \mathcal{F}_{M \sqcup N}^{(H)} S^N, Z]^G & \xrightarrow{\cong} & [\mathcal{F}_{M \sqcup N}^{(H)} S^N, Z]^H & \xrightarrow{\cong} & \pi_0(\mathrm{map}_H(S^N, Z(M \sqcup N))) \end{array}$$

The right vertical map is a bijection since Z is a $G^{\mathcal{U}}\Omega$ -spectrum. \square

The notion of $G^{\mathcal{U}}$ -stable equivalence behaves well under geometric realization and singular complex:

Proposition 2.47 (Relation between spaces and simplicial sets). *Both geometric realization $|\cdot|$ and singular complex \mathcal{S} preserve and reflect $G^{\mathcal{U}}$ -stable equivalences.*

Proof. This follows from the fact that the geometric realization and singular complex adjunction induces an equivalence on level homotopy categories and that a G -symmetric spectrum of spaces is a $G^{\mathcal{U}}\Omega$ -spectrum if and only if its singular complex is. \square

3 Naive homotopy groups, $G^{\mathcal{U}}$ -semistability and the action of $M_G^{\mathcal{U}}$

In this section we deal with “naive” equivariant homotopy groups of G -symmetric spectra. This part is the main complication in the theory of G -symmetric spectra, because unlike for G -orthogonal spectra a $G^{\mathcal{U}}$ -stable equivalence must not necessarily induce isomorphisms on these, hence the name “naive”. Nevertheless they are useful for two reasons: Firstly, the converse is true, i.e., every map inducing an isomorphism on naive homotopy groups is a $G^{\mathcal{U}}$ -stable equivalence (Theorem 3.48) and for a lot of maps it is easier to show that they induce such an isomorphism than to check the rather abstract condition in the definition of a $G^{\mathcal{U}}$ -stable equivalence.

Secondly, there is a large class of G -symmetric spectra, called $G^{\mathcal{U}}$ -semistable, on which the two notions of equivalence agree. For non-equivariant symmetric spectra several equivalent characterizations of semistability were given in [HSS00, Section 5.6]. This theory was developed further in [Sch08], where it is shown that the naive homotopy groups carry a natural action of the monoid of injective self-maps of the natural numbers and that a symmetric spectrum is semistable if and only if this action is trivial. These criteria have equivariant analogs, in particular there exists a tame action of the monoid of G -equivariant injective self-maps of the chosen G -set universe \mathcal{U} on the naive homotopy groups of G -symmetric spectra which detects whether the naive homotopy groups are isomorphic to the derived ones (Corollary 3.46). The notion of tameness and the relevant algebraic properties of such actions are given in Section 3.3.

Moreover, in Section 3.5 we explain that the naive equivariant homotopy groups of G -symmetric spectra naturally form a $G^{\mathcal{U}}$ -Mackey functor, the construction being very similar to the one for homotopy groups of G -orthogonal spectra. Furthermore, this $G^{\mathcal{U}}$ -Mackey functor structure is compatible with the above monoid action.

3.1 Definition

We begin by defining the naive equivariant homotopy groups of a G -symmetric spectrum and fix a G -set universe \mathcal{U} . We denote by $s_G(\mathcal{U})$ the poset of finite G -subsets of \mathcal{U} , partially ordered by inclusion.

Definition 3.1. Let n be an integer and $H \leq G$ a subgroup. The n -th H -equivariant homotopy group $\pi_n^{H, \mathcal{U}} X$ of a G -symmetric spectrum of spaces X (with respect to \mathcal{U}) is defined as

$$\pi_n^{H, \mathcal{U}} X := \operatorname{colim}_{M \in s_G(\mathcal{U})} [S^{n \sqcup M}, X(M)]^H.$$

The connecting maps in the colimit system are given by the composites

$$[S^{n \sqcup M}, X(M)]^H \xrightarrow{(-) \wedge S^{N-M}} [S^{n \sqcup M \sqcup (N-M)}, X(M) \wedge S^{N-M}]^H \xrightarrow{(\sigma_M^{N-M})^*} [S^{n \sqcup N}, X(N)]^H$$

for every inclusion $M \subseteq N$. The last step implicitly uses the homeomorphism $X(M \sqcup (N-M)) \cong X(N)$ induced from the canonical isomorphism $M \sqcup (N-M) \cong N$.

To clarify what this exactly means for negative n we choose an isometric G -embedding $i : \mathbb{R}^\infty \rightarrow (\mathbb{R}[\mathcal{U}])^G$ and only index the colimit system over those elements M of $s_H(\mathcal{U})$ such that \mathbb{R}^M contains $i(\mathbb{R}^{-n})$, in which case the corresponding term is given by $[S^{M-i(\mathbb{R}^{-n})}, X(M)]^H$, the expression $M - i(\mathbb{R}^{-n})$ denoting the orthogonal complement of $i(\mathbb{R}^{-n})$ in \mathbb{R}^M . Since any two such embeddings are connected by a homotopy (of embeddings), in fact the space of embeddings is contractible, the definition only depends on this choice up to *canonical* isomorphism and so we leave it out of the notation.

If the number of G -orbits of M is larger than $-n$ by at least two, the permutation sphere $S^{n \sqcup M}$ has at least two trivial coordinates and hence the set $[S^{n \sqcup M}, X(M)]^H$ has a natural abelian group structure. Since such G -sets are cofinal in $s_G(\mathcal{U})$, this turns $\pi_0^{H, \mathcal{U}} X$ into an abelian group. For a G -symmetric spectrum of simplicial sets A we set $\pi_n^{H, \mathcal{U}} A := \pi_n^{H, \mathcal{U}} |A|$.

Example 3.2 (Homotopy groups of $G^\mathcal{U}\Omega$ -spectra). For every (positive) $G^\mathcal{U}\Omega$ -spectrum of spaces X , every integer n , every subgroup H of G and every finite (non-empty) G -set $M \subseteq \mathcal{U}$, for which $i(\mathbb{R}^{-n}) \subseteq \mathbb{R}[M]$ if n is negative, the induced map $[S^{n \sqcup M}, X(M)]^H \rightarrow \pi_n^{H, \mathcal{U}} X$ is a bijection.

Proof. This follows from the fact that by adjunction the maps in the colimit system are equal to the composite

$$[S^{n \sqcup M}, X(M)]^H \xrightarrow{(\tilde{\sigma}_M^{N-M})_*} [S^{n \sqcup M}, \Omega^{M-N} X(M \sqcup (N - M))]^H \xrightarrow{\cong} [S^{n \sqcup N}, X(N)]^H.$$

For (positive) $G^\mathcal{U}\Omega$ -spectra the map $\tilde{\sigma}_M^{N-M}$ is a genuine G -equivalence, hence by the Whitehead theorem (Proposition 1.19) it induces bijections on homotopy classes of G -maps out of G -CW complexes and thus in particular out of representation spheres. So all the terms above the one for M in the (directed) colimit system are bijections and it follows that the map to the colimit is an isomorphism. \square

Definition 3.3 ($\underline{\pi}_*^\mathcal{U}$ -isomorphism). A map $f : X \rightarrow Y$ of G -symmetric spectra is called a $\underline{\pi}_*^\mathcal{U}$ -isomorphism if for every subgroup H of G and every integer $n \in \mathbb{Z}$ the induced map

$$\pi_n^{H, \mathcal{U}}(f) : \pi_n^{H, \mathcal{U}}(X) \rightarrow \pi_n^{H, \mathcal{U}}(Y)$$

is an isomorphism.

A (positive) $G^\mathcal{U}$ -level equivalence induces a bijection on (almost) all terms in the colimit system and so:

Lemma 3.4. *Every (positive) $G^\mathcal{U}$ -level equivalence is a $\underline{\pi}_*^\mathcal{U}$ -isomorphism.*

Remark 3.5. The n -th homotopy group $\pi_n^{H, \mathbb{R}[\mathcal{U}]} X$ of a G -orthogonal spectrum X with respect to the G -representation universe $\mathbb{R}[\mathcal{U}]$ is defined similarly by forming a colimit over $[S^{n+V}, X(V)]^H$ for all finite dimensional G -subrepresentations of $\mathbb{R}[\mathcal{U}]$. Since subrepresentations of the form $\mathbb{R}[M]$ for a finite G -subset M of \mathcal{U} are cofinal in that system, Lemma 2.13 implies that $\pi_n^{H, \mathbb{R}[\mathcal{U}]} X$ is naturally isomorphic to $\pi_n^{H, \mathcal{U}}(UX)$.

3.2 Suspension isomorphism, long exact sequences and the Wirthmüller isomorphism

In this section we explain how naive homotopy groups behave under various constructions. We begin with the suspension isomorphism:

Proposition 3.6. *For every G -symmetric spectrum of spaces X and every finite dimensional G -subrepresentation V of $\mathbb{R}[\mathcal{U}]$ the adjunction unit $X \xrightarrow{\eta_X} \Omega^V(S^V \wedge X)$ and counit $\epsilon_X : S^V \wedge (\Omega^V X) \rightarrow X$ are $\underline{\pi}_*^\mathcal{U}$ -isomorphisms.*

Proof. We start with the unit. By adjunction, we have to prove that the suspension maps $S^V \wedge (-) : [S^{n \sqcup M}, X(M)]^G \rightarrow [S^V \wedge S^{n \sqcup M}, S^V \wedge X(M)]^G$ induce an isomorphism on the colimit over $s_G(\mathcal{U})$. For later use we now show the stronger statement that the maps $S^V \wedge (-) : [A \wedge S^{M-W}, X(M)]^G \rightarrow [S^V \wedge A \wedge S^{M-W}, S^V \wedge X(M)]^G$ always induce an isomorphism on the colimit for any based G -space A and finite dimensional G -subrepresentation $W \subseteq \mathbb{R}[\mathcal{U}]$, where

S^{M-W} is again defined as the sphere of the orthogonal complement of W in $\mathbb{R}[M]$ (and hence the indexing system is restricted to those M for which W is contained in $\mathbb{R}[M]$).

We first show that the induced map on colimits is injective. For this we take an element in the kernel, represented by a G -map $f : A \wedge S^{M-W} \rightarrow X(M)$. Replacing f by a representative higher up in the colimit system if necessary, we can assume that the suspension $S^V \wedge f$ is already G -nullhomotopic. By assumption, there is a finite G -set N contained in \mathcal{U} (and disjoint from M) and an embedding $j : V \rightarrow \mathbb{R}^N$. Then it follows that $f \wedge S^N = f \wedge S^{j(V)} \wedge S^{N-j(V)}$ is also G -nullhomotopic and hence f already represents the zero class in the domain colimit.

For surjectivity we let M be a finite G -subset of \mathcal{U} (such that its linearization contains W) and $f : S^V \wedge A \wedge S^{M-W} \rightarrow S^V \wedge X(M)$ a G -map. Without loss of generality we can take M to be disjoint to the N chosen above. We denote by g the G -map $A \wedge S^{M-W} \wedge S^V \rightarrow X(M) \wedge S^V$ obtained by pre- and postcomposing f with the symmetry isomorphisms shifting S^V to the correct position. Then the map $S^V \wedge g$ differs from $f \wedge S^V$ (as maps $S^V \wedge A \wedge S^{M-W} \wedge S^V \rightarrow S^V \wedge X(M) \wedge S^V$) by pre- and postcomposing with the twist automorphism of $S^V \wedge S^V$. In general, these twists do not cancel each other out and so $S^V \wedge g$ is not always homotopic to $f \wedge S^V$. However, they do become homotopic after smashing with another copy of $S^V \wedge S^V$, as a consequence of the following lemma:

Lemma 3.7. *If two permutations in Σ_4 have the same sign, their actions on $(S^V)^{\wedge 4}$ are equivariantly homotopic.*

Proof of Lemma. Writing $(S^V)^{\wedge 4}$ as $S^{V \otimes \mathbb{R}^4}$ shows that the Σ_4 -action extends to a continuous $O(4)$ action which commutes with the G -action. Any two permutations with the same sign are connected by a path in $O(4)$, so their actions are homotopic. \square

Hence, up to homotopy, $(S^V \wedge g) \wedge S^V \wedge S^V$ differs from $(f \wedge S^V) \wedge S^V \wedge S^V$ by pre- and postcomposing with the twist map of the two S^V factors on the right, which cancel each other out. It follows that the element represented by

$$\sigma_M^N \circ (g \wedge S^{N-j(V)}) : A \wedge S^{(M-W) \sqcup N} = A \wedge S^{M-W} \wedge S^V \wedge S^{N-j(V)} \rightarrow X(M \sqcup N)$$

is an inverse image of the class of f , and so the induced map is surjective and in particular the unit is a $\pi_*^{\mathcal{U}}$ -isomorphism (after applying it for all subgroups with the restricted action on V).

Regarding the counit ϵ_X , we note that the composition

$$\operatorname{colim}_M [S^{n \sqcup (M-V)}, \Omega^V X(M)]^G \xrightarrow{S^V \wedge (-)} \pi_n^{G, \mathcal{U}}(S^V \wedge (\Omega^V X)) \xrightarrow{(\epsilon_X)_*} \pi_n^{G, \mathcal{U}} X$$

equals the isomorphism adjoining S^V to the left (and hence canceling out the negative copy of V). Applying what we just showed for W equal to V and A equal to S^n (or W equal to $V \oplus \mathbb{R}^{-n}$ and A equal to S^0 , if n is negative) and the spectrum $\Omega^V X$, we see that the first map is also an isomorphism and thus ϵ_X induces an isomorphism on $\pi_n^{G, \mathcal{U}}$. By considering the restricted representation, it follows that both the unit and counit also induce isomorphisms on $\pi_n^{H, \mathcal{U}}$ for proper subgroups H of G , which finishes the proof. \square

Since for any G -set universe \mathcal{U} the linearization $\mathbb{R}[\mathcal{U}]$ always allows an embedding of the trivial \mathbb{R}^∞ (even if no trivial G -sets embed into \mathcal{U}), the above proposition in particular applies for all trivial representations. By adjunction, the groups $\pi_n^{H, \mathcal{U}}(\Omega(S^1 \wedge X))$ can be naturally identified with $\pi_{n+1}^{H, \mathcal{U}}(S^1 \wedge X)$ and so we see that there is a natural isomorphism $\pi_n^{H, \mathcal{U}} X \cong \pi_{n+1}^{H, \mathcal{U}}(S^1 \wedge X)$ for all subgroups H of G and $n \in \mathbb{Z}$.

One uses this to construct long exact sequences in naive homotopy groups. The mapping cone $C(f)$ and the homotopy fiber $H(f)$ of a map $f : X \rightarrow Y$ of G -symmetric spectra (as well as the associated natural maps $i(f) : Y \rightarrow C(f)$, $q(f) : C(f) \rightarrow S^1 \wedge X$, $j(f) : \Omega Y \rightarrow H(f)$ and $p(f) : H(f) \rightarrow X$) are defined levelwise.

Proposition 3.8 (Long exact sequences in homotopy groups). *For every map $f : X \rightarrow Y$ of G -symmetric spectra of spaces and all subgroups H of G the sequences*

$$\dots \rightarrow \pi_n^{H,\mathcal{U}} X \xrightarrow{\pi_n^{H,\mathcal{U}} f} \pi_n^{H,\mathcal{U}} Y \xrightarrow{\pi_n^{H,\mathcal{U}} i(f)} \pi_n^{H,\mathcal{U}} C(f) \xrightarrow{\pi_n^{H,\mathcal{U}} q(f)} \pi_n^{H,\mathcal{U}} (S^1 \wedge X) \cong \pi_{n-1}^{H,\mathcal{U}} X \rightarrow \dots$$

and

$$\dots \rightarrow \pi_{n+1}^{H,\mathcal{U}} Y \cong \pi_{n+1}^{H,\mathcal{U}} (\Omega Y) \xrightarrow{\pi_{n+1}^{H,\mathcal{U}} j(f)} \pi_{n+1}^{H,\mathcal{U}} H(f) \xrightarrow{\pi_{n+1}^{H,\mathcal{U}} p(f)} \pi_{n+1}^{H,\mathcal{U}} X \xrightarrow{\pi_{n+1}^{H,\mathcal{U}} f} \pi_{n+1}^{H,\mathcal{U}} Y \rightarrow \dots$$

are exact.

Proof. The proof is very similar to the non-equivariant one. To show exactness of the first sequence one mainly uses that if a G -map $g : S^{n \sqcup M} \rightarrow Y(M)$ becomes nullhomotopic after postcomposing with $i(f)(M)$, its one-fold suspension $g \wedge S^1$ factors over $f(M) \wedge S^1 : X(M) \wedge S^1 \rightarrow Y(M) \wedge S^1$ up to G -homotopy. Exactness of the second sequence follows from the levelwise sequences $\dots \rightarrow [S^{n \sqcup M}, H(f)(M)]^G \rightarrow [S^{n \sqcup M}, X(M)]^G \rightarrow [S^{n \sqcup M}, Y(M)]^G \rightarrow \dots$ already being exact. \square

The first sequence above is also exact for G -symmetric spectra of simplicial sets, since geometric realization commutes with mapping cones. The second one is exact if X and Y are $G^{\mathcal{U}}$ -projective level fibrant.

From these exact sequences one obtains the following corollaries:

Corollary 3.9. *Let $f : X \rightarrow Y$ be a map of G -symmetric spectra of spaces. Then the natural map $h : S^1 \wedge H(f) \rightarrow C(f)$ is a $\pi_*^{\mathcal{U}}$ -isomorphism.*

Proof. This follows from the comparison of the two long exact sequences associated to f . The composite $\pi_n^{H,\mathcal{U}} H(f) \xrightarrow{\cong} \pi_{n+1}^{H,\mathcal{U}} (S^1 \wedge H(f)) \xrightarrow{h_*} \pi_{n+1}^{H,\mathcal{U}} C(f)$ makes all the squares commute up to a sign and hence one can apply the five-lemma to deduce that it is a $\pi_*^{\mathcal{U}}$ -isomorphism. \square

In the simplicial case there still exists a natural zig-zag of $\pi_*^{\mathcal{U}}$ -isomorphisms from $S^1 \wedge H(f)$ to $C(f)$, provided that X and Y are $G^{\mathcal{U}}$ -projective level fibrant.

Corollary 3.10 (Homotopy groups of wedges and products). *Let $\{X_i\}_{i \in I}$ be a family of G -symmetric spectra. Then for every $n \in \mathbb{Z}$ and every subgroup H of G the natural map $\bigoplus_{i \in I} \pi_n^{H,\mathcal{U}} X_i \rightarrow \pi_n^{H,\mathcal{U}} \left(\bigvee_{i \in I} X_i \right)$ is an isomorphism. Furthermore, for finite I the natural map $\pi_n^{H,\mathcal{U}} \left(\prod_{i \in I} X_i \right) \rightarrow \prod_{i \in I} \pi_n^{H,\mathcal{U}} X_i$ is an isomorphism and hence the canonical morphism $\bigvee_{i \in I} X_i \rightarrow \prod_{i \in I} X_i$ is a $\pi_*^{\mathcal{U}}$ -isomorphism.*

Proof. The statement about wedges is proved by induction for finite I , using that the cone of the inclusion $X_1 \rightarrow X_1 \vee X_2$ is G -homotopy equivalent to X_2 . Writing I as the colimit of its finite subsets and using the fact that homotopy groups commute with directed colimits along closed inclusions gives the general case.

The isomorphism for finite products of G -symmetric spectra follows from the fact that colimits commute with finite products of abelian groups. \square

Now we come to the Wirthmüller isomorphism, which states that for a subgroup inclusion $H \leq G$ the natural map γ from induction to coinduction (cf. Section 1.1) is a $\pi_*^{\mathcal{U}}$ -isomorphism, provided that G/H allows an embedding into $\mathbb{R}[\mathcal{U}]$. It was first proved for suspension spectra in [Wir74], generalized to all H -spectra in [LMS86, Section II.6] and reproved in a different way in [May03]. The statement in [LMS86] and [May03] is about the derived natural transformation in the G -equivariant stable homotopy category. In the case of a complete G -universe an undervived

version is given in [Sch11, Theorem 5.22], the main point being that one does not have to replace the H -orthogonal spectrum X cofibrantly for the isomorphism to hold.

The version for G -symmetric spectra we present here is even more underived, as it involves naive homotopy groups and not the true, derived ones:

Proposition 3.11 (Wirthmüller isomorphism). *Let $H \leq G$ be a subgroup inclusion such that G/H allows an embedding into $\mathbb{R}[\mathcal{U}]$ and X an H -symmetric spectrum of spaces or simplicial sets. Then the natural map $\gamma_X : G \ltimes_H X \rightarrow \text{map}_H(G, X)$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism.*

We start with a lemma:

Lemma 3.12. *Let H and K be subgroups of G , A a cofibrant H -space, N a finite G -set for which there exists a G -embedding $G/H \hookrightarrow \mathbb{R}[N]$ and k a natural number. Then:*

$$\text{conn}(\gamma_{A \wedge S^{kN}}^K) \geq \min(2 \text{conn}((S^{kN})^K) + 1, \text{conn}((S^{kN})^K) + k)$$

Proof. We use the decompositions

$$\text{res}_K^G(G \ltimes_H (A \wedge S^{kN})) \cong \bigvee_{[g] \in K \backslash G/H} K \ltimes_{K \cap gHg^{-1}} (c_g^*(\text{res}_{g^{-1}Kg \cap H}^G(A \wedge S^{kN}))) \quad (3.1)$$

and

$$\text{res}_K^G(\text{map}_H(G, A \wedge S^{kN})) \cong \prod_{[g] \in K \backslash G/H} \text{map}_{K \cap gHg^{-1}}(K, c_g^*(\text{res}_{g^{-1}Kg \cap H}^G(A \wedge S^{kN}))). \quad (3.2)$$

described in Section 1.1 to determine the K -fixed points on both sides. We start with the domain: If K is not contained in gHg^{-1} , the summand associated to g is a proper induction and hence has trivial K -fixed points. If it is contained, the K -fixed points of that summand are homeomorphic to the gKg^{-1} -fixed points of $A \wedge S^{kN}$.

The K -fixed points of a factor $\text{map}_{K \cap gHg^{-1}}(K, c_g^*(\text{res}_{g^{-1}Kg \cap H}^G(A \wedge S^{kN})))$ are given by the $K \cap gHg^{-1}$ fixed points of $c_g^*(\text{res}_{g^{-1}Kg \cap H}^G(A \wedge S^{kN}))$ which in turn are homeomorphic to the $gKg^{-1} \cap H$ -fixed points of $A \wedge S^{kN}$. Building that information in, we see that on K -fixed points the map $\gamma_{A \wedge S^{kN}}$ becomes the composite

$$\bigvee_{\substack{[g] \in K \backslash G/H \\ K \subseteq gHg^{-1}}} (A \wedge S^{kN})^{gKg^{-1}} \rightarrow \prod_{\substack{[g] \in K \backslash G/H \\ K \subseteq gHg^{-1}}} (A \wedge S^{kN})^{gKg^{-1}} \rightarrow \prod_{[g] \in K \backslash G/H} (A \wedge S^{kN})^{gKg^{-1} \cap H}$$

Hence, the connectivity is greater than or equal to the minimum of the connectivities of the constituents, and so it remains to determine them. Using that all the terms are non-equivariantly cofibrant, we see that the inclusion of the wedge into the product has connectivity at least as large as the sum of the two smallest connectivities of the individual terms plus 1. But the connectivity of $(A \wedge S^{kN})^{gKg^{-1}}$ is at least as large as the connectivity of $(S^{kN})^{gKg^{-1}}$, which is equal to the connectivity of $(S^{kN})^K$, so we obtain $2 \text{conn}((S^{kN})^K) + 1$ as a lower bound for the connectivity of the first map. The inclusion of the factors into the larger product is as connective as the lowest connectivity of an added factor. Since G/H allows an embedding into $\mathbb{R}[N]$, we see that there is an element of S^N whose isotropy is exactly H . This implies that if $g^{-1}Kg$ is not contained in H , the $g^{-1}Kg \cap H$ -fixed points of $\mathbb{R}[N]$ are at least one dimension larger than the $g^{-1}Kg$ -fixed points, which are isomorphic to those of K . Hence $\text{conn}((A \wedge S^{kN})^{g^{-1}Kg \cap H}) \geq \text{conn}((S^{kN})^K) + k$ and we are done. \square

We now prove Proposition 3.11 in the case where all evaluations of X at H -representations are cofibrant H -spaces. This already implies the simplicial version.

Proof. By the long exact sequence (Proposition 3.8) it suffices to show that the equivariant homotopy groups of the homotopy fiber $H(\gamma_X)$ are trivial. For this we let $f : S^{n \sqcup M} \rightarrow H(\gamma_X)(M) = H(\gamma_{X(M)})$ be a representative of an element in $\pi_n^{G, \mathcal{U}} H(\gamma_X)$. For another finite G -subset of \mathcal{U} of the form kN and disjoint to M , the structure map $H(\gamma_{X(M)}) \wedge S^{kN} \rightarrow H(\gamma_{X(M \sqcup kN)})$ factors through $H(\gamma_{X(M) \wedge S^{kN}})$. Now, for a subgroup K of G the dimension of the K -fixed points of $S^{n \sqcup M} \wedge S^{kN}$ is given by $n + \dim(M^K) + \dim(kN^K) = n + \dim(M^K) + \text{conn}(kN^K) + 1$. Choosing N so that it allows an embedding of G/H , Lemma 3.2 implies that the connectivity of $H(\gamma_{X(M) \wedge S^{kN}})$ is at least the minimum of the two numbers $2 \text{conn}((S^{kN})^K) + 1$ and $\text{conn}((S^{kN})^K) + k$. For k sufficiently large these two numbers are bigger than the dimension of $(S^{n \sqcup M} \wedge S^{kN})^K$. Hence, if we take the maximum of all the k 's required for different subgroups K , we see that if we suspend high enough there cannot be any homotopically non-trivial maps from $S^{n \sqcup M} \wedge S^{kN}$ into $H(\gamma_{X(M) \wedge S^{kN}})$, in particular the suspension $f \wedge S^{kN}$ becomes nullhomotopic when composed with the map $H(\gamma_{X(M)}) \wedge S^{kN} \rightarrow H(\gamma_{X(M) \wedge S^{kN}})$ and hence f represents the zero class in the colimit. Thus $\pi_n^{G, \mathcal{U}} H(\gamma_X)$ is trivial for all n and γ_X induces an isomorphism on $\pi_*^{G, \mathcal{U}}$. To show that it also induces an isomorphism on $\pi_*^{H, \mathcal{U}}$ for proper subgroups K of G one uses the double coset decompositions 3.1 and 3.2 and Corollary 3.10. \square

To obtain the result for arbitrary H -symmetric spectra it suffices to show that a G -flat replacement $X^b \rightarrow X$ induces $\underline{\pi}_*^{\mathcal{U}}$ -isomorphisms $\text{map}_H(G, X^b) \rightarrow \text{map}_H(G, X)$ and $G \rtimes_H X^b \rightarrow G \rtimes_H X$. The prior follows from the natural isomorphism $\pi_*^{G, \mathcal{U}}(\text{map}_H(G, X)) \cong \pi_*^{H, \mathcal{U}} X$ and Decomposition 3.2, so it remains to show the latter. This is a consequence of the following lemma, where we do not require that G/H embeds into $\mathbb{R}[\mathcal{U}]$:

Lemma 3.13. *The functor $G \rtimes_H -$ maps $\underline{\pi}_*^{\mathcal{U}}$ -isomorphisms of H -symmetric spectra to $\underline{\pi}_*^{\mathcal{U}}$ -isomorphisms of G -symmetric spectra.*

Proof of lemma. Since $G \rtimes_H -$ preserves cofiber sequences, the long exact sequence of Proposition 3.8 implies that it suffices to show that if an H -symmetric spectrum X has trivial homotopy groups, then so does $G \rtimes_H X$. We prove this by induction on the order of G , the induction start for the trivial group being clear. Hence we take a finite group G , assume the statement shown for all proper subgroups and fix an H -symmetric spectrum X with trivial homotopy groups. Now from Decomposition 3.1 and the formula for the homotopy groups of a wedge we see that all the groups $\pi_n^{K, \mathcal{U}}(G \rtimes_H X)$, where K is a proper subgroup, are trivial by the induction hypothesis. So let $f : S^{n \sqcup M} \rightarrow G \rtimes_H X(M)$ be a G -map, we have to show that it represents the trivial element in $\pi_n^{G, \mathcal{U}}(G \rtimes_H X)$. If H is equal to G the statement of the Lemma is trivial, so we can assume this not to be the case. But then the G -fixed points of $G \rtimes_H X(M)$ only consist of the basepoint and hence f factors through a map $\tilde{f} : S^{n \sqcup M} / (S^{n \sqcup M})^G \rightarrow G \rtimes_H X(M)$. Now the domain of \tilde{f} is a finite based G -CW complex with all cells induced, and an induction over the cells shows that any such map into a level of $G \rtimes_H X$ is stably trivial, since the groups $\pi_*^{K, \mathcal{U}}(G \rtimes_H X)$ vanish for proper subgroups. This finishes the proof. \square

Hence, the Wirthmüller isomorphism is proven in all cases. We obtain a corollary:

Corollary 3.14. *Let \mathcal{F} be a family of subgroups of G , $f : X \rightarrow Y$ a map of G -symmetric spectra which induces an isomorphism on $\pi_n^{H, \mathcal{U}}$ for all $n \in \mathbb{Z}$ and all H in \mathcal{F} and A a cofibrant G -space with non-basepoint isotropy contained in \mathcal{F} . Then*

- (i) $A \wedge f : A \wedge X \rightarrow A \wedge Y$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism.
- (ii) If A is finite and X and Y are $G^{\mathcal{U}}$ -projective level fibrant, $\text{map}(A, f) : \text{map}(A, X) \rightarrow \text{map}(A, Y)$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism.

Proof. We first show the statements for finite I_G -cell complexes A . By an induction over the cells, making use of the long exact sequence for the mapping cone and homotopy fiber, both statements reduce to the case where A is of the form $G/H_+ \wedge S^k$ for some $k \in \mathbb{N}$ and H in \mathcal{F} . Smashing with/mapping out of S^k only shifts the homotopy groups, so it remains to show the case $A = G/H_+$. Since there are natural isomorphisms $G/H_+ \wedge X \cong G \ltimes_H \text{res}_H^G X$ and $\text{map}(G/H, X) \cong \text{map}_H(G, \text{res}_H^G X)$, this follows from Lemma 3.13 and the preceding paragraph.

In order to obtain statement (i) also for infinite A one uses that sequential colimits along G -flat cofibrations preserve $\pi_*^{\mathcal{U}}$ -isomorphisms. Finally, the statements follow for retracts of I_G -cell complexes by functoriality. \square

In particular, taking \mathcal{F} to be the family of all subgroups, we see that smashing with any cofibrant G -space preserves all $\pi_*^{\mathcal{U}}$ -isomorphisms, and so does $\text{map}(A, -)$ for finite A (with the fibrancy assumption above in the simplicial case).

Finally we obtain the following extension of Proposition 3.6:

Proposition 3.15. *For every G -subrepresentation V of $\mathbb{R}[\mathcal{U}]$ the functors $S^V \wedge (-)$ and Ω^V preserve and reflect $\pi_*^{\mathcal{U}}$ -isomorphisms of G -symmetric spectra of spaces. Furthermore, a map $f : S^V \wedge X \rightarrow Y$ is a $\pi_*^{\mathcal{U}}$ -isomorphism if and only if its adjoint $\tilde{f} : X \rightarrow \Omega^V Y$ is.*

Proof. Since representation spheres allow the structure of a G -CW complex, we already know that $S^V \wedge -$ and Ω^V preserve $\pi_*^{\mathcal{U}}$ -isomorphisms by the previous corollary. Both the unit and counit of the adjunction $(S^V \wedge (-), \Omega^V)$ are $\pi_*^{\mathcal{U}}$ -isomorphisms, so we see that both functors also reflect $\pi_*^{\mathcal{U}}$ -isomorphisms. Finally, the adjoint \tilde{f} of a map $f : S^V \wedge X \rightarrow Y$ is given by the composite $X \xrightarrow{\eta_X} \Omega^V(S^V \wedge X) \xrightarrow{\Omega^V(f)} Y$ and so it is a $\pi_*^{\mathcal{U}}$ -isomorphism if and only if f is. \square

3.3 The monoid ring $\mathbf{M}_G^{\mathcal{U}}$

The universe \mathcal{U} is kept in the notation of $\pi_*^{G, \mathcal{U}}$ for two reasons: For once, as one would expect and as is also the case for G -orthogonal spectra, these homotopy groups depend on the isomorphism type of \mathcal{U} , for example taking \mathcal{U} to be a trivial universe will usually lead to different homotopy groups than for a complete universe. Secondly, if two universes $\mathcal{U}, \mathcal{U}'$ are isomorphic, any such isomorphism $\varphi : \mathcal{U} \cong \mathcal{U}'$ induces a natural isomorphism $\varphi_* : \pi_n^{G, \mathcal{U}} \cong \pi_n^{G, \mathcal{U}'}$, but this isomorphism does depend on the chosen φ . (We note, however, that the notion of $\pi_*^{\mathcal{U}}$ -isomorphism only depends on the isomorphism type of \mathcal{U} .) This phenomenon, which already occurs for non-equivariant symmetric spectra, is not present for G -orthogonal spectra. It is encoded in an action of the monoid ring $\mathbf{M}_G^{\mathcal{U}}$, for which we now introduce the relevant algebraic background. Everything here is a rather straightforward generalization of [Sch08].

Definition 3.16 (Monoid of equivariant self-injections). The set $\text{Inj}_G(\mathcal{U}, \mathcal{U})$ of G -equivariant injective self-maps of \mathcal{U} forms a monoid under composition. We denote the associated monoid ring $\mathbb{Z}[\text{Inj}_G(\mathcal{U}, \mathcal{U})]$ by $\mathbf{M}_G^{\mathcal{U}}$.

Let M be a finite G -subset of \mathcal{U} and A an $\mathbf{M}_G^{\mathcal{U}}$ -module. Then we say that an element a of A is of *filtration* M if every injection ψ which leaves M pointwise fixed acts trivially on a .

Definition 3.17 (Tame modules). An $\mathbf{M}_G^{\mathcal{U}}$ -module A is called *tame* if every element $a \in A$ is of filtration M for some finite G -subset M of \mathcal{U} .

Tame modules have the following property:

Lemma 3.18. *Let A be a tame $\mathbf{M}_G^{\mathcal{U}}$ -module. Then the following hold:*

- (i) *If two injections $\psi, \psi' : \mathcal{U} \rightarrow \mathcal{U}$ agree on a finite G -subset M of \mathcal{U} , then $\psi \cdot a = \psi' \cdot a$ for every element $a \in A$ of filtration M .*

(ii) Every element $\psi \in \text{Inj}_G(\mathcal{U}, \mathcal{U})$ acts injectively on A .

Proof. (i): Let $\alpha : \mathcal{U} \cong \mathcal{U}$ be a G -bijection which agrees with ψ and ψ' on M . Then $\alpha^{-1} \circ \psi$ and $\alpha^{-1} \circ \psi'$ restrict to the identity on M and thus

$$\psi \cdot a = \alpha \cdot ((\alpha^{-1} \circ \psi) \cdot a) = \alpha \cdot a = \alpha((\alpha^{-1} \circ \psi') \cdot a) = \psi' \cdot a$$

for every element a of filtration M .

Regarding (ii), let ψ be an injective self-map of \mathcal{U} and a an element of A such that $\psi \cdot a = 0$. Since A is tame, the element a is of filtration M for some finite G -subset $M \in \mathcal{U}$. Let $\alpha : \mathcal{U} \rightarrow \mathcal{U}$ be a G -bijection which agrees with ψ on M . Then, by (i), we have $\alpha \cdot a = 0$ and thus $a = \alpha^{-1} \cdot (\alpha \cdot a) = \alpha^{-1} \cdot 0 = 0$ and hence ψ acts injectively. \square

An example of an $\mathbf{M}_G^\mathcal{U}$ -module which is not tame is the free module of rank one. A class of non-trivial examples of tame $\mathbf{M}_G^\mathcal{U}$ -modules is given by the following:

Example 3.19. Let M be a finite G -set. Then the abelian group $\mathcal{P}(M, \mathcal{U}) = \mathbb{Z}[\text{Inj}_G(M, \mathcal{U})]$ (which is non-trivial if and only if M allows an embedding into \mathcal{U}) becomes an $\mathbf{M}_G^\mathcal{U}$ -module via postcomposition. Since every injection takes image in a finite G -subset of \mathcal{U} , it is tame.

All G -embeddings $\varphi : \mathcal{U} \hookrightarrow \mathcal{U}$ give rise to a conjugation ring homomorphism $c_\varphi : \mathbf{M}_G^\mathcal{U} \rightarrow \mathbf{M}_G^\mathcal{U}$, even if they are not surjective:

Definition 3.20 (Conjugation). The φ -conjugate of a G -embedding $f : \mathcal{U} \hookrightarrow \mathcal{U}$ is defined as:

$$c_\varphi(f)(x) = \begin{cases} \varphi(f(\varphi^{-1}(x))) & \text{if } x \in \text{im}(\varphi) \\ x & \text{if } x \notin \text{im}(\varphi) \end{cases}$$

Given an $\mathbf{M}_G^\mathcal{U}$ -module A we denote by $c_\varphi^* A$ the same abelian group with $\mathbf{M}_G^\mathcal{U}$ -action pulled back along c_φ . With this definition the map $A \xrightarrow{\varphi} c_\varphi^* A$ becomes $\mathbf{M}_G^\mathcal{U}$ -equivariant.

Any preimage $\varphi^{-1}(x)$ is unique if it exists, so the above is well-defined.

Definition 3.21 (Shift). Let M be a finite G -subset of \mathcal{U} . Then a *shift by M* is a G -embedding $d^M : \mathcal{U} \hookrightarrow \mathcal{U}$ with image the complement $(\mathcal{U} - M)$ of M .

The behavior of these shifts along finite G -sets is central to the theory of semistability (cf. Section 3.6). The non-equivariant prototype is the map $d : \mathbb{N} \rightarrow \mathbb{N}$ from [Sch08, Lemma 2.1, (iii)] which sends i to $i + 1$. Equivariantly, shifts in every isotypical direction contained in \mathcal{U} are needed. Since we prefer to work coordinate-free there is no canonical “shifting back by one copy of M ”, leading to the ambiguity in the definition above. However, any two shifts by M only differ by precomposition with a unique G -automorphism.

Remark 3.22. Let M and N be finite G -subsets of \mathcal{U} , d^M a shift by M and d^N a shift by N . Then the composition $d^M \circ d^N$ is a shift by $M \sqcup d^M(N)$.

By iteration one obtains a shift along the whole universe \mathcal{U} . For this we choose a sequence $\emptyset = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots \subseteq \mathcal{U}$ of finite G -subsets of \mathcal{U} whose union equals \mathcal{U} , along with an $(M_n - M_{n-1})$ -shift $d^{M_n - M_{n-1}}$ leaving M_{n-1} fixed (for all $n \in \mathbb{N}$). Then the composite $d^{M_n} := d^{M_n - M_{n-1}} \circ d^{M_{n-1} - M_{n-2}} \circ \dots \circ d^{M_1}$ is a shift by M_n . Given an $\mathbf{M}_G^\mathcal{U}$ -module A we define a new $\mathbf{M}_G^\mathcal{U}$ -module $c_{d^\mathcal{U}}^* A$ as the colimit of the sequence

$$A \xrightarrow{d^{M_1}} c_{d^{M_1}}^* A \xrightarrow{d^{M_2 - M_1}} c_{d^{M_2}}^* A \rightarrow \dots \rightarrow c_{d^{M_n}}^* A \rightarrow \dots$$

and denote the induced map $A \rightarrow c_{d^\mathcal{U}}^* A$ by $d^\mathcal{U}$.

We then have:

Proposition 3.23 (Criteria for triviality). *For a tame $\mathbf{M}_G^{\mathcal{U}}$ -module A the following are equivalent:*

- (i) *The $\mathbf{M}_G^{\mathcal{U}}$ -action on A is trivial.*
- (ii) *All shifts along transitive G -subsets of \mathcal{U} act surjectively on A .*
- (iii) *All shifts along finite G -subsets of \mathcal{U} act surjectively on A .*
- (iv) *The map $d^{\mathcal{U}} : A \rightarrow c_{d^{\mathcal{U}}}^* A$ above is surjective.*
- (v) *There exists a finite G -subset M of \mathcal{U} such that every element of A is of filtration M .*

Proof. For a (possibly countably infinite) sequence of injective maps the total composite is a surjection if and only if each of the constituents is. Together with the uniqueness of shifts up to precomposition with an automorphism and Remark 3.22 this shows that conditions (ii), (iii) and (iv) are equivalent. Moreover, it is immediate that (i) implies all the others.

The remaining implications are a consequence of:

Lemma 3.24. *If an element a of A does not have filtration \emptyset (i.e., if there exists an injection $\mathcal{U} \hookrightarrow \mathcal{U}$ which acts non-trivially on a) and $d^M : \mathcal{U} \hookrightarrow \mathcal{U}$ is a shift by M , then $d^M \cdot a$ is not of filtration M .*

Proof. Let $\psi : \mathcal{U} \hookrightarrow \mathcal{U}$ be an injection with $\psi \cdot a \neq a$. Then, by the injectivity of $d^M \cdot -$ we have

$$d^M \cdot a \neq d^M \cdot (\psi \cdot a) = c_{d^M}(\psi) \cdot (d^M \cdot a)$$

and $c_{d^M}(\psi)$ is an injection which pointwise fixes M . Hence, $d^M \cdot a$ is not of filtration M . \square

Now this implies that if an element a is of filtration M and not \emptyset , it cannot be in the image of $d^M \cdot -$ for any shift by M (because a preimage b would need to be of filtration \emptyset by the lemma, implying in particular that $a = d^M \cdot b = b$, which contradicts the fact that a is not of filtration \emptyset). Hence, (iii) implies (i).

Finally, (v) implies (i), because if there existed an element a not of filtration \emptyset , then $d^M \cdot a$ would not be of filtration M by the lemma, contradicting the assumption. \square

Corollary 3.25. *Every tame $\mathbf{M}_G^{\mathcal{U}}$ -module which is finitely generated as an abelian group has trivial action.*

Proof. Let A be generated by a_1, \dots, a_n . Then each a_i is of filtration M_i for some finite G -subset M_i of \mathcal{U} , hence they are all of filtration $\cup M_i$, which is again finite. It follows that every element of A is of filtration $\cup M_i$, so by item (v) in the previous proposition the action is trivial. \square

3.4 Action of $\mathbf{M}_G^{\mathcal{U}}$ on naive homotopy groups of G -symmetric spectra

Now we introduce the action of $\mathbf{M}_G^{\mathcal{U}}$ on the naive homotopy groups of a G -symmetric spectrum. It suffices to explain it in the case $n = 0$, since $\pi_n^{H, \mathcal{U}} X$ is canonically isomorphic to $\pi_0^{H, \mathcal{U}}(\Omega^n X)$ for $n > 0$ and to $\pi_0^{H, \mathcal{U}}(S^{-n} \wedge X)$ for $n < 0$ (cf. Proposition 3.6). So let $\alpha : \mathcal{U} \hookrightarrow \mathcal{U}$ be a G -equivariant injection and x an element of $\pi_0^{H, \mathcal{U}} X$. Then x is represented by an H -equivariant map $f : S^M \rightarrow X(M)$ for some finite G -subset M of \mathcal{U} .

Definition 3.26. The element $\alpha \cdot x \in \pi_n^{H, \mathcal{U}} X$ is defined as the class of the composite

$$\alpha_* f : S^{\alpha(M)} \xrightarrow{S^{\alpha|_M}{}^{-1}} S^M \xrightarrow{f} X(M) \xrightarrow{X(\alpha|_M)} X(\alpha(M)),$$

i.e., the injection monoid $\mathbf{M}_G^{\mathcal{U}}$ acts “through conjugation”.

Remark 3.27. The construction also makes sense if α is an injective G -equivariant map between two different (and possibly not even isomorphic) G -set universes \mathcal{U} and \mathcal{U}' , inducing a map $\alpha \cdot (-) : \pi_n^{H, \mathcal{U}} X \rightarrow \pi_n^{H, \mathcal{U}'} X$.

In order to see that this is well-defined we let N be another finite G -subset of \mathcal{U} disjoint to M and consider the commutative diagram

$$\begin{array}{ccccccc}
S^{\alpha(M)} \wedge S^{\alpha(N)} & \xrightarrow{\alpha_{|M}^{-1} \wedge \alpha_{|N}^{-1}} & S^M \wedge S^N & \xrightarrow{f \wedge S^N} & X(M) \wedge S^N & \xrightarrow{X(\alpha_{|M}) \wedge S^{\alpha_{|N}}} & X(\alpha(M)) \wedge S^{\alpha(N)} \\
\cong \downarrow & & \cong \downarrow & & \downarrow \sigma_M^N & & \downarrow \sigma_{\alpha(M)}^{\alpha(N)} \\
S^{\alpha(M \sqcup N)} & \xrightarrow{\alpha_{|M \sqcup N}^{-1}} & S^{M \sqcup N} & \longrightarrow & X(M \sqcup N) & \xrightarrow{X(\alpha_{|M \sqcup N})} & X(\alpha(M \sqcup N))
\end{array}$$

where the commutativity of the lower square is a consequence of the equivariance of the structure maps with respect to products of symmetric groups. So we see that

$$\sigma_{\alpha(M)}^{\alpha(N)} \circ (\alpha_* f \wedge S^{\alpha(N)}) = \alpha_*(\sigma_M^N \circ (f \wedge S^N))$$

and hence $\alpha_* f$ and $\alpha_*(\sigma_M^N \circ (f \wedge S^N))$ define the same class in $\pi_0^{H, \mathcal{U}} X$.

It is straightforward to check that the action is unital, associative, additive and natural. Furthermore, every element in $\pi_0^{H, \mathcal{U}} X$ represented by a map $f : S^M \rightarrow X(M)$ is of filtration M in the sense of the previous section, since by definition the action of an injection α on $[f]$ depends only on the restriction of α to M . In particular, this implies that $\pi_0^{H, \mathcal{U}} X$ is a tame $\mathbf{M}_G^{\mathcal{U}}$ -module.

Remark 3.28. In order for an injection $\alpha : \mathcal{U} \hookrightarrow \mathcal{U}$ to act on $\pi_n^{H, \mathcal{U}} X$ it would suffice that it is H -equivariant and not necessarily G -equivariant. In fact, it is possible to show that the multiplicative coefficient system $G/H \mapsto \mathbb{Z}[\text{Inj}_H(\mathcal{U}, \mathcal{U})]$ naturally acts on the $G^{\mathcal{U}}$ -Mackey functor $\underline{\pi}_n^{\mathcal{U}} X$. However, since finite G -subsets are cofinal in the poset of finite H -subsets of \mathcal{U} , the action of this coefficient system is trivial if and only if the one of $\mathbf{M}_G^{\mathcal{U}}$ is (cf. Proposition 3.23), and hence the latter is enough to detect semistability. So we restrict ourselves to $\mathbf{M}_G^{\mathcal{U}}$ -actions to avoid further complication.

Definition 3.29 (Semistability). A G -symmetric spectrum X is called $G^{\mathcal{U}}$ -semistable if the action of $\mathbf{M}_G^{\mathcal{U}}$ on $\pi_n^{H, \mathcal{U}} X$ is trivial for every subgroup H of G and every $n \in \mathbb{Z}$.

We have:

Lemma 3.30. *Every $G^{\mathcal{U}}$ -spectrum is $G^{\mathcal{U}}$ -semistable.*

Proof. Let X be a $G^{\mathcal{U}}$ -spectrum and $n \in \mathbb{Z}$ an integer. By Example 3.2 there exists a finite G -subset M of \mathcal{U} such that the map $[S^{n \sqcup M}, X(M)]^H \rightarrow \pi_n^{H, \mathcal{U}} X$ is a bijection. In particular, it is surjective and hence every element in $\pi_n^{H, \mathcal{U}} X$ is of filtration M . By criterion (v) of Proposition 3.23, this implies that the action of $\mathbf{M}_G^{\mathcal{U}}$ is trivial. \square

This already implies that if a G -symmetric spectrum X is not $G^{\mathcal{U}}$ -semistable, there cannot exist a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism from X to a $G^{\mathcal{U}}$ -spectrum. Conversely, in Corollary 3.43 and Proposition 3.45 it is shown that every $G^{\mathcal{U}}$ -semistable G -symmetric spectrum admits such a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism. Example 3.35 explains that the free spectrum $\mathcal{F}_M S^M$ is not $G^{\mathcal{U}}$ -semistable for any non-empty finite G -set M .

Finally, we note the following immediate consequence of Corollary 3.25:

Corollary 3.31. *Every G -symmetric spectrum X for which all $\pi_n^{H, \mathcal{U}} X$ are finitely generated as abelian groups is $G^{\mathcal{U}}$ -semistable.*

3.5 Mackey functor structure

In this section we show that for every $n \in \mathbb{Z}$ the collection $\pi_n^{\mathcal{U}} X = \{\pi_n^{H, \mathcal{U}} X\}_{H \leq G}$ of n -th naive homotopy groups of a G -symmetric spectrum X naturally forms a $G^{\mathcal{U}}$ -Mackey functor in tame $\mathbf{M}_G^{\mathcal{U}}$ -modules. Here, the chosen G -set universe \mathcal{U} affects the functoriality (cf. [Lew95, Def. 1.5]), as we now explain. We start with the definition of a $G^{\mathcal{U}}$ -Mackey functor in abelian groups. For this we let $g(\mathcal{U})$ be the set of inclusions $H \leq K$ of subgroups of G for which there exists a K -embedding $K/H \hookrightarrow \mathbb{R}[\mathcal{U}]$ (not necessarily into $\mathcal{U}!$). We note that $g(\mathcal{U})$ is closed under conjugation with all elements of G and under composition of inclusions.

Definition 3.32. A $G^{\mathcal{U}}$ -Mackey functor A consists of

- an abelian group $A(H)$ for every subgroup H of G ,
- contravariantly functorial restriction homomorphisms $\text{res}_H^K : A(K) \rightarrow A(H)$ for every subgroup inclusion $H \leq K$,
- covariantly functorial transfer homomorphisms $\text{tr}_H^K : A(H) \rightarrow A(K)$ for every inclusion $H \leq K$ contained in $g(\mathcal{U})$
- and transitive conjugation homomorphisms $c_g : A(H) \rightarrow A(gHg^{-1})$ for every element $g \in G$ and subgroup H of G .

This data has to satisfy three conditions:

- Inner automorphisms act trivially, i.e., for all subgroups H and every element h of H the conjugation homomorphism $c_h : A(H) \rightarrow A(H)$ is equal to the identity.
- The restriction and transfer homomorphisms have to be compatible with the conjugation homomorphisms in the sense that $c_g \circ \text{res}_H^K = \text{res}_{gHg^{-1}}^{gKg^{-1}} \circ c_g$ and $c_g \circ \text{tr}_H^K = \text{tr}_{gHg^{-1}}^{gKg^{-1}} \circ c_g$.
- The restriction and transfer homomorphisms have to satisfy the *double coset formula*, i.e., for every pair of subgroup inclusions $H \leq K$ and $J \leq K$, of which the former is required to lie in $g(\mathcal{U})$, the equation

$$\text{res}_J^K \circ \text{tr}_H^K = \sum_{[g] \in J \backslash K / H} \text{tr}_{J \cap (gHg^{-1})}^J \circ c_g \circ \text{res}_{(g^{-1}Jg) \cap H}^H$$

holds, as maps from $A(H)$ to $A(J)$.

If $\mathbb{R}[\mathcal{U}]$ is a complete G -representation universe, there are covariantly functorial transfer homomorphisms for every subgroup inclusion and a $G^{\mathcal{U}}$ -Mackey functor is usually simply called a G -Mackey functor. If \mathcal{U} is the trivial G -set universe, a $G^{\mathcal{U}}$ -Mackey functor is merely a coefficient system, i.e., a contravariant functor from the orbit category of G to abelian groups.

Definition 3.33. A $G^{\mathcal{U}}$ -Mackey functor in tame $\mathbf{M}_G^{\mathcal{U}}$ -modules is a $G^{\mathcal{U}}$ -Mackey functor A in abelian groups where the values $A(H)$ are all given the structure of a tame $\mathbf{M}_G^{\mathcal{U}}$ -module and restriction, transfer and conjugation maps are equivariant with respect to this action.

We now explain that the naive homotopy groups of a G -symmetric spectrum have this structure. So we let X be a G -symmetric spectrum. For a subgroup H of G we set $(\pi_n^{\mathcal{U}} X)(H) = \pi_n^{H, \mathcal{U}} X$ with the tame $\mathbf{M}_G^{\mathcal{U}}$ -module structure introduced in the previous section. The construction of the three kinds of structure maps above is very similar to that of G -orthogonal spectra, cf. [Sch11, Section 3]. Furthermore, we can restrict ourselves to the case $n = 0$ by replacing X by $\Omega^n X$ for positive n or $S^{-n} \wedge X$ for negative n .

Restriction maps: Let $H \leq K \leq G$ be a subgroup inclusion and $x \in \pi_0^{K,\mathcal{U}} X$ an element represented by a K -map $f : S^M \rightarrow X(M)$ for some finite G -subset M of \mathcal{U} . Then f is in particular H -equivariant for the restricted H -action, so it also represents an element of $\pi_0^{H,\mathcal{U}} X$ and we set $\text{res}_H^K(x) = \text{res}_H^K([f]) = [f]$ to be this element. Clearly this construction is independent of the choice of f (and hence well-defined), contravariantly functorial in subgroup inclusions and commutes with the $\mathbf{M}_G^\mathcal{U}$ -action.

Transfer maps: Let $H \leq K \leq G$ be an inclusion contained in $g(\mathcal{U})$ and $x \in \pi_0^{H,\mathcal{U}} X$ an element represented by an H -map $f : S^M \rightarrow X(M)$ for a finite K -set $M \subseteq \mathcal{U}$. By assumption, there exists a K -embedding $K/H \hookrightarrow \mathcal{U}$. Enlarging M if necessary we can further assume that this embedding takes image in $\mathbb{R}[M]$. By choosing small balls around each point in the image one can extend this to an embedding $K \ltimes_H D(\mathbb{R}[M]) \rightarrow \mathbb{R}[M]$, where $D(\mathbb{R}[M])$ denotes the unit disc inside $\mathbb{R}[M]$. Quotienting out by all the points outside the image of this embedding, one obtains a based Pontryagin-Thom collapse map $p_H^K : S^M \rightarrow K \ltimes_H S^M$. The transfer $\text{tr}_H^K(x)$ is then defined to be the class of the composite

$$S^M \xrightarrow{p_H^K} K \ltimes_H S^M \xrightarrow{K \ltimes_H f} K \ltimes_H X(M) \xrightarrow{\mu} X(M),$$

where μ denotes the multiplication map, using that $X(M)$ is a K -space (in fact a G -space).

To see that this construction does not depend on the embedding $K/H \hookrightarrow \mathcal{U}$ one uses that the space of all such embeddings is connected (and in fact contractible). It is then straightforward to see that it also commutes with suspensions along structure maps and is hence independent of the choice of f . Covariant functoriality follows from the fact that given two inclusions $J \leq H$, $H \leq K$ and embeddings $K/H \xrightarrow{K} \mathbb{R}[M]$ and $H/J \xrightarrow{H} \mathbb{R}[M]$ the composite

$$S^M \xrightarrow{p_H^K} K \ltimes_H S^M \xrightarrow{K \ltimes_H p_J^H} K \ltimes_H (H \ltimes_J S^M) \cong K \ltimes_J S^M$$

is a Thom-Pontryagin collapse for the associated embedding $K/J \hookrightarrow K \ltimes_H \mathbb{R}[M] \hookrightarrow \mathbb{R}[M]$ (where the latter is the restriction of the embedding $K \ltimes_H D(\mathbb{R}[M]) \hookrightarrow \mathbb{R}[M]$ above to the interior of $D(\mathbb{R}[M])$).

Given a G -equivariant injection $\alpha : \mathcal{U} \hookrightarrow \mathcal{U}$, the composition

$$S^{\alpha(M)} \xrightarrow{S^{\alpha|_M}{}^{-1}} S^M \xrightarrow{p_H^K} K \ltimes_H S^M \xrightarrow{K \ltimes_H S^{\alpha|_M}} K \ltimes_H S^{\alpha(M)}$$

is a Thom-Pontryagin collapse for the embedding $K/H \hookrightarrow \mathbb{R}[M] \xrightarrow{\mathbb{R}[\alpha]} \mathbb{R}[\alpha(M)]$, so it follows that the transfer is $\mathbf{M}_G^\mathcal{U}$ -equivariant.

Conjugation maps: Let $g \in G$ be any element and $x \in \pi_0^{H,\mathcal{U}} X$ be represented by the H -map $f : S^M \rightarrow X(M)$ for some finite G -subset M of \mathcal{U} , i.e., an H -fixed point of the G -space $\text{map}(S^M, X(M))$. Then the composite $S^M \xrightarrow{g \cdot} S^M \xrightarrow{f} X(M) \xrightarrow{g \cdot} X(M)$ is gHg^{-1} -equivariant, since $ghg^{-1}(fgg^{-1})gh^{-1}g^{-1} = ghfh^{-1}g^{-1} = gfg^{-1}$. So we can set

$$c_g(x) = c_g([f]) = [gfg^{-1}].$$

It is straightforward to check that this is independent of the choice of f , covariantly functorial, compatible with restriction and transfers in the sense above and $\mathbf{M}_G^\mathcal{U}$ -equivariant. Furthermore, inner automorphisms act as the identity, because if $f : S^M \rightarrow X(M)$ is H -equivariant, conjugation with any element h of H leaves it unchanged.

We omit the proof that the double coset formula is satisfied, as it is similar to the one for G -orthogonal spectra and lengthy (a detailed proof can be found in [Sch13, Prop. 4.21]). It is not needed elsewhere in this paper.

Remark 3.34. The compatibility of the $\mathbf{M}_G^{\mathcal{U}}$ -action with the $G^{\mathcal{U}}$ -Mackey functor structure can be useful for proving $G^{\mathcal{U}}$ -semistability. Namely, if an element $x \in \pi_0^{H,\mathcal{U}} X$ is acted upon trivially by all G -injections $\mathcal{U} \hookrightarrow \mathcal{U}$, then the same is true for all restrictions $\text{res}_K^H(x)$, transfers $\text{tr}_H^L(x)$ and conjugations $c_g(x)$ of x .

Now we can give an example of a G -symmetric spectrum which is not $G^{\mathcal{U}}$ -semistable:

Example 3.35 (Free spectra). Let M be a finite G -set not necessarily contained in \mathcal{U} and $\mathcal{F}_M S^M$ the free spectrum in level M on the permutation sphere S^M (cf. Definition 2.18). Then the N -th level is given by $(\mathcal{F}_M S^M)(N) = S^M \wedge \Sigma(M, N)$. This is G -isomorphic to $\text{Inj}(M, N)_+ \wedge S^N$ (with diagonal G -action) via the map sending a tuple $((x \in S^M) \wedge (\alpha : M \rightarrow N, y \in S^{N-\alpha(M)})$ to $(\alpha \wedge (\alpha(x) \wedge y))$. Hence we see that

$$\begin{aligned} \pi_0^{H,\mathcal{U}}(\mathcal{F}_M S^M) &= \text{colim}_{N \in s(\mathcal{U})} [S^N, \text{Inj}(M, N)_+ \wedge S^N]^H \cong \text{colim}_{N \in s(\mathcal{U})} [S^N, \text{Inj}(M, \mathcal{U})_+ \wedge S^N]^H \\ &= \pi_0^{H,\mathcal{U}}(\Sigma^\infty \text{Inj}(M, \mathcal{U})_+) \end{aligned}$$

with $\mathbf{M}_G^{\mathcal{U}}$ -action via post-composition on the discrete space $\text{Inj}(M, \mathcal{U})_+$. It follows from the Segal-tom Dieck splitting (which originally appeared in [tD75, Satz 2], or is explained in more modern terms [Sch11, Thm. 6.12]) that the $G^{\mathcal{U}}$ -Mackey functor $\underline{\pi}_0^{\mathcal{U}}(\Sigma^\infty \text{Inj}(M, \mathcal{U})_+)$ is isomorphic to the free $G^{\mathcal{U}}$ -Mackey functor on the coefficient system $H \mapsto \mathbb{Z}[\text{Inj}(M, \mathcal{U})^H] = \mathbb{Z}[\text{Inj}_H(M, \mathcal{U})]$. Concretely, this means that $\pi_0^{H,\mathcal{U}}(\Sigma^\infty(\text{Inj}(M, \mathcal{U})_+))$ (and hence $\pi_0^{H,\mathcal{U}}(\mathcal{F}_M S^M)$) is a free abelian group with basis $\{\text{tr}_K^H(\alpha)\}_{(K,\alpha) \in R}$ where R runs through a system of representatives of H -conjugacy classes of pairs (K, α) , for which K is a subgroup of H such that H/K allows an embedding into $\mathbb{R}[\mathcal{U}]$ and α is a K -equivariant inclusion $M \hookrightarrow \mathcal{U}$. The monoid ring $\mathbf{M}_G^{\mathcal{U}}$ acts through α and permutes this basis. If M is not the empty set, this action is not trivial and so $\mathcal{F}_M S^M$ is not $G^{\mathcal{U}}$ -semistable. If M is contained in \mathcal{U} the $\mathbf{M}_G^{\mathcal{U}}$ -module $\pi_0^{H,\mathcal{U}}(\mathcal{F}_M S^M)$, contains the module $\mathcal{P}(M, \mathcal{U})$ of Example 3.19 as a submodule, as the span of precisely those basis elements which are not a transfer from a proper subgroup.

An interesting special case is the following: We let \mathcal{U} be a countably infinite free G -set and take M to be $\underline{1}$ with trivial G -action. Then there are no H -equivariant injections $\underline{1} \rightarrow \mathcal{U}$ for any non-trivial subgroup H and hence $\pi_0^{H,\mathcal{U}}(\mathcal{F}_1 S^1)$ has a basis $\{\text{tr}_e^H(\alpha)\}$ where α runs through a set of representatives of H -conjugacy classes of injections $\underline{1} \rightarrow \mathcal{U}$, or in other words through the set of H -orbits of \mathcal{U} . In particular, every element of $\pi_0^{H,\mathcal{U}}(\mathcal{F}_1 S^1)$ is a transfer of an element of $\pi_0^{e,\mathcal{U}}(\mathcal{F}_1 S^1)$. This is not the case if one replaces \mathcal{U} by a complete G -set universe \mathcal{U}' , since any element of the G -fixed points $(\mathcal{U}')^G$ gives a basis element of $\pi_0^{G,\mathcal{U}'}(\mathcal{F}_1 S^1)$ which is not a transfer from a proper subgroup. This shows that even if two G -set universes \mathcal{U} and \mathcal{U}' satisfy $\mathbb{R}[\mathcal{U}] \cong \mathbb{R}[\mathcal{U}']$ (and in particular the notions of $G^{\mathcal{U}}$ - and $G^{\mathcal{U}'}$ -Mackey functor agree), it is not true that $\underline{\pi}_0^{\mathcal{U}} X$ and $\underline{\pi}_0^{\mathcal{U}'} X$ are isomorphic for all G -symmetric spectra X .

This example also shows that the $G^{\mathcal{U}}$ -stable equivalence $\lambda_{\emptyset, M} : \mathcal{F}_M S^M \rightarrow \mathbb{S}$ of Example 2.46 is not a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism if M is non-empty.

3.6 Shift and relation to $\mathbf{M}_G^{\mathcal{U}}$ -action

We now discuss the effect of the shift sh^M along a finite G -set M (contained in \mathcal{U}) on homotopy groups, or more precisely the effect of the composite $\Omega^M sh^M$ together with the natural transformation $\tilde{\alpha}^M : id \rightarrow \Omega^M sh^M$ of Definition 2.16. By adjunction, the n -th homotopy group $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X)$ is naturally isomorphic to the colimit over the terms $[S^{n \sqcup M \sqcup N}, X(M \sqcup N)]^H$ for all finite G -subsets N of \mathcal{U} . Finite G -subsets of the form $M \sqcup N$ are cofinal in the G -set universe $M \sqcup \mathcal{U}$, so we see that this colimit and hence $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X)$ is naturally isomorphic to $\pi_n^{H, M \sqcup \mathcal{U}} X$. Since $M \sqcup \mathcal{U}$ is isomorphic to \mathcal{U} , it follows that the homotopy groups $\pi_n^{H,\mathcal{U}} X$ and $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X)$ are abstractly isomorphic. This already shows:

Corollary 3.36. *The shift sh^M along any finite G -subset $M \subseteq \mathcal{U}$ preserves and reflects $\pi_*^{\mathcal{U}}$ -isomorphisms.*

Proof. By the (non-canonical) isomorphism of homotopy groups above, a map $f : X \rightarrow Y$ is a $\pi_*^{\mathcal{U}}$ -isomorphism if and only if $\Omega^M sh^M f$ is. The latter is equivalent to $sh^M f$ being one by Proposition 3.15. \square

However, the fact that the isomorphism $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \cong \pi_n^{H,\mathcal{U}} X$ is not canonical (it depends on a choice of isomorphism $M \sqcup \mathcal{U} \cong \mathcal{U}$) should make one skeptical about whether such an isomorphism is induced by $\tilde{\alpha}_X^M$. In fact, while there is no canonical isomorphism $M \sqcup \mathcal{U} \cong \mathcal{U}$, there is of course a canonical G -equivariant embedding $i : \mathcal{U} \hookrightarrow M \sqcup \mathcal{U}$ and it turns out that this describes the effect of $\tilde{\alpha}_X^M$ on homotopy groups:

Proposition 3.37. *The composite $\pi_n^{H,\mathcal{U}} X \xrightarrow{(\tilde{\alpha}_X^M)_*} \pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \cong \pi_n^{H,M \sqcup \mathcal{U}} X$ equals the action of the inclusion $i : \mathcal{U} \hookrightarrow M \sqcup \mathcal{U}$, in the sense of Remark 3.27.*

Proof. Let $x \in \pi_n^{H,\mathcal{U}} X$ be an arbitrary element, represented by an H -map $f : S^{n \sqcup N} \rightarrow X(N)$ for some finite H -set N contained in \mathcal{U} . Then $i_* f$ is again f , with the only difference of thinking of N as now sitting inside $M \sqcup \mathcal{U}$. In order to compare it to an element of $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X)$ under the isomorphism above, we have to suspend this element by M , i.e., form the composition

$$S^{n \sqcup N} \wedge S^M \xrightarrow{f \wedge S^M} X(N) \wedge S^M \xrightarrow{\sigma_N^M} X(N \sqcup M) \xrightarrow{X(\chi_{N,M})} X(M \sqcup N).$$

But, by the definition of $\tilde{\alpha}_X^M$, after adjoining S^M to the right this is precisely the composition

$$S^{n \sqcup N} \xrightarrow{f} X(N) \xrightarrow{(\tilde{\alpha}_X^M)(N)} \Omega^M X(M \sqcup N) = (\Omega^M sh^M X)(N)$$

and thus $i \cdot x = [i_* f] = [\tilde{\alpha}_X^M(N) \circ f] = \tilde{\alpha}_X^M(x)$. \square

This proposition can be translated into a statement about the internal action of $\mathbf{M}_G^{\mathcal{U}}$ on $\pi_n^{H,\mathcal{U}} X$, where it corresponds to the algebraic shift discussed in Definition 3.21. For this we choose a “shift by M ”, i.e., an embedding $d^M : \mathcal{U} \hookrightarrow \mathcal{U}$ with image $\mathcal{U} - M$. In particular we obtain an isomorphism $id_M \sqcup d^M : M \sqcup \mathcal{U} \xrightarrow{\cong} \mathcal{U}$ of G -set universes and hence a natural isomorphism $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \cong \pi_n^{H,M \sqcup \mathcal{U}} X \cong \pi_n^{H,\mathcal{U}} X$. This isomorphism is not $\mathbf{M}_G^{\mathcal{U}}$ -equivariant in general, but it becomes so when conjugating the action on $\pi_n^{H,\mathcal{U}} X$ along d^M in the sense of Definition 3.20:

Proposition 3.38. *The above defines an $\mathbf{M}_G^{\mathcal{U}}$ -isomorphism*

$$\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \cong c_{d^M}^*(\pi_n^{H,\mathcal{U}} X).$$

Moreover, the composition $\pi_n^{H,\mathcal{U}} X \xrightarrow{\tilde{\alpha}_X^M} \pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \xrightarrow{\cong} c_{d^M}^(\pi_n^{H,\mathcal{U}} X)$ equals multiplication with d^M .*

Proof. Under the isomorphism $\pi_n^{H,\mathcal{U}}(\Omega^M sh^M X) \cong \pi_n^{H,M \sqcup \mathcal{U}} X$ the $\mathbf{M}_G^{\mathcal{U}}$ -action on the former corresponds to the $\mathbf{M}_G^{M \sqcup \mathcal{U}}$ -action on the latter pulled back along the homomorphism $id_M \sqcup - : \mathbf{M}_G^{\mathcal{U}} \rightarrow \mathbf{M}_G^{M \sqcup \mathcal{U}}$, since M is left untouched. The equality $(id_M \sqcup d^M) \circ i_{\mathcal{U}} = d^M$ implies that after conjugation with $id_M \sqcup d^M$ this homomorphism becomes c_{d^M} , and so the action on $\pi_n^{H,\mathcal{U}} X$ is pulled back along c_{d^M} as claimed. The second statement is then an immediate consequence of Proposition 3.37. \square

Applying Lemma 3.18 and Proposition 3.23 on algebraic properties of tame $\mathbf{M}_G^{\mathcal{U}}$ -modules, we obtain the following corollaries:

Corollary 3.39. *The map $\tilde{\alpha}_X^M : X \rightarrow (\Omega^M sh^M X)$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism if and only if d^M (and hence every shift by M) acts surjectively on $\pi_n^{H,\mathcal{U}} X$ for all subgroups H of G and all $n \in \mathbb{Z}$.*

Corollary 3.40. *A G -symmetric spectrum X is $G^{\mathcal{U}}$ -semistable if and only if the map $\tilde{\alpha}_X^M : X \rightarrow \Omega^M sh^M X$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism for every finite G -subset M of \mathcal{U} and if and only if $\alpha_X^M : S^M \wedge X \rightarrow sh^M X$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism for every finite G -subset M of \mathcal{U} .*

The last “if and only if” follows from Proposition 3.15.

Corollary 3.41. *Let \mathcal{F} be a family of subgroups of G , A a cofibrant G -space with non-basepoint isotropy in \mathcal{F} and X a G -symmetric spectrum such that $\mathbf{M}_G^{\mathcal{U}}$ acts trivially on $\pi_*^{H,\mathcal{U}} X$ for all H in \mathcal{F} . Then $A \wedge X$ is $G^{\mathcal{U}}$ -semistable. If A is finite and X is $G^{\mathcal{U}}$ -projective level fibrant, then $\text{map}(A, X)$ is also $G^{\mathcal{U}}$ -semistable.*

Proof. This follows from Corollaries 3.40 and 3.14, since $A \wedge (-)$ commutes both with $S^M \wedge (-)$ and $sh^M(-)$ and $\text{map}(A, -)$ commutes with $\Omega^M sh^M(-)$. \square

Corollary 3.42. *If X is $G^{\mathcal{U}}$ -semistable, then so is $sh^N X$ for any finite G -subset N of \mathcal{U} .*

Proof. The shift $sh^N(-)$ commutes up to natural isomorphism with $sh^M(-)$ and $S^M \wedge (-)$, so this also follows from Corollaries 3.40 and 3.36. \square

We now introduce the endofunctor $\Omega^{\mathcal{U}} sh^{\mathcal{U}}$ on G -symmetric spectra (together with a natural transformation $\tilde{\alpha}^{\mathcal{U}} : id \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}}$), the equivariant analog of the construction called R^{∞} in [HSS00, Theorem 3.1.11]. It serves two purposes for us: First, we show that for all $G^{\mathcal{U}}$ -semistable G -symmetric spectra X the G -symmetric spectrum $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is a $G^{\mathcal{U}} \Omega$ spectrum and the map $\tilde{\alpha}_X^{\mathcal{U}}$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism, proving that $G^{\mathcal{U}}$ -semistable G -symmetric spectra can be replaced by $G^{\mathcal{U}} \Omega$ -spectra up to $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism. Secondly, via an equivariant version of the proof of [HSS00, Theorem 3.1.11], it is used to show that every $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism is a $G^{\mathcal{U}}$ -stable equivalence.

We choose an exhaustive filtration $\emptyset = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq \dots \subseteq \mathcal{U}$ of \mathcal{U} by finite G -subsets. By that we mean that every M_n is a finite G -subset of \mathcal{U} and the union of all M_n ’s equals \mathcal{U} . For a G -symmetric spectrum of spaces X we then define $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ as the mapping telescope of the sequence

$$X = \Omega^{M_0} sh^{M_0} X \rightarrow \Omega^{M_1} sh^{M_1} X \rightarrow \Omega^{M_2} sh^{M_2} X \rightarrow \dots$$

with induced natural map $\tilde{\alpha}_X^{\mathcal{U}} : X \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}} X$. The connecting maps in the system are given by

$$\Omega^{M_n} sh^{M_n} X \xrightarrow{\Omega^{M_n} \tilde{\alpha}_{sh^{M_n} X}^{(M_{n+1}-M_n)}} \Omega^{M_n} \Omega^{M_{n+1}-M_n} sh^{M_{n+1}-M_n} sh^{M_n} X \cong \Omega^{M_{n+1}} sh^{M_{n+1}} X.$$

The effect on homotopy groups of this construction corresponds to the infinite shift below Definition 3.21 (applied degreewise to $\mathbf{M}_G^{\mathcal{U}}$ -Mackey functors), whose terminology we now use. So for every $n \in \mathbb{N}$ we choose an $(M_n - M_{n-1})$ -shift $d^M : \mathcal{U} \hookrightarrow \mathcal{U}$ leaving M_{n-1} fixed. By composition we obtain M_n -shifts d^{M_n} , inductively defined via $d^M = d^{M_n - M_{n-1}} \circ d^{M_{n-1}}$. Making use of Proposition 3.38 we see that there is an isomorphism of sequences

$$\begin{array}{ccccccc} \underline{\pi}_n^{\mathcal{U}} X & \longrightarrow & \underline{\pi}_n^{\mathcal{U}} (\Omega^{M_1} sh^{M_1} X) & \longrightarrow & \underline{\pi}_n^{\mathcal{U}} (\Omega^{M_2} sh^{M_2} X) & \longrightarrow & \dots \\ \downarrow id & & \downarrow \cong & & \downarrow \cong & & \\ \underline{\pi}_n^{\mathcal{U}} X & \xrightarrow{d^{M_1}} & c_{d^{M_1}}^* (\underline{\pi}_n^{\mathcal{U}} X) & \xrightarrow{d^{M_2-M_1}} & c_{d^{M_2}}^* (\underline{\pi}_n^{\mathcal{U}} X) & \longrightarrow & \dots \end{array}$$

of $\mathbf{M}_G^{\mathcal{U}}$ -modules. In particular, it follows that the map $(\tilde{\alpha}_X^{\mathcal{U}})_* : \pi_n^{H,\mathcal{U}} X \rightarrow \pi_n^{H,\mathcal{U}} (\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)$ can be identified with $d^{\mathcal{U}} : \pi_n^{H,\mathcal{U}} X \rightarrow c_{d^{\mathcal{U}}}^* (\pi_n^{H,\mathcal{U}} X)$ and via Proposition 3.23 we obtain:

Corollary 3.43. *The map $\tilde{\alpha}_X^{\mathcal{U}} : X \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is a $\pi_*^{\mathcal{U}}$ -isomorphism if and only if X is $G^{\mathcal{U}}$ -semistable.*

Now we want to show that if X is $G^{\mathcal{U}}$ -semistable, then $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is a $G^{\mathcal{U}}\Omega$ -spectrum. For this we need:

Lemma 3.44. *(i) For every G -symmetric spectrum of spaces X , every subgroup H of G and every two finite H -sets L and M there is a natural bijection*

$$[S^L, (\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M)]^H \cong \pi_0^{H, \mathcal{U}}(\Omega^L sh^M X)$$

Moreover, given another finite H -subset N of \mathcal{U} disjoint to M , the square

$$\begin{array}{ccc} \pi_k((\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M)^H) & \xrightarrow{(\tilde{\sigma}_M^N)^*} & \pi_k((\Omega^N(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M))^H) \\ \cong \downarrow & & \downarrow \cong \\ \pi_k^{H, \mathcal{U}}(sh^M X) & \xrightarrow{(\tilde{\alpha}_{sh^M X}^N)^*} & \pi_k^{H, \mathcal{U}}(\Omega^N sh^{M \sqcup N} X) \end{array}$$

commutes, where the vertical isomorphisms are those obtained by setting $L = k$ respectively $L = N \sqcup k$.

(ii) For every $G^{\mathcal{U}}\Omega$ -spectrum of spaces X , the map $\tilde{\alpha}_X^{\mathcal{U}} : X \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is a $G^{\mathcal{U}}$ -level equivalence.

We note that the spectrum $\Omega^M sh^M X$ in the lemma is in general no longer a G -symmetric spectrum, but only an H -symmetric spectrum.

Proof. Regarding (i), we see:

$$\begin{aligned} [S^L, (\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M)]^H &\cong \operatorname{colim}_{n \in \mathbb{N}} [S^L, (\Omega^{M_n} sh^{M_n} X)(M)]^H \cong \operatorname{colim}_{n \in \mathbb{N}} [S^{L \sqcup M_n}, (sh^{M_n} X)(M)]^H \\ &\cong \operatorname{colim}_{n \in \mathbb{N}} [S^{L \sqcup M_n}, X(M_n \sqcup M)]^H \cong \operatorname{colim}_{n \in \mathbb{N}} [S^{L \sqcup M_n}, X(M \sqcup M_n)]^H \\ &\cong \pi_0^{H, \mathcal{U}}(\Omega^L(sh^M X)) \end{aligned}$$

The last step uses that the M_n are an exhaustive sequence in \mathcal{U} . The described levelwise isomorphisms are compatible with the maps in the colimit system since on both sides neither L nor M are touched by the stabilization maps. The commutativity of the diagram is a direct check from the definitions.

By Corollary 2.42, for every $G^{\mathcal{U}}\Omega$ -spectrum Z and every finite G -set $M \subseteq \mathcal{U}$ the map $\alpha_Z^M : Z \rightarrow \Omega^M sh^M Z$ is a $G^{\mathcal{U}}$ -level equivalence and $\Omega^M sh^M Z$ is again a $G^{\mathcal{U}}\Omega$ -spectrum. Since looping with respect to a G -set preserves $G^{\mathcal{U}}$ -level equivalences, it follows that every map in the sequence is a $G^{\mathcal{U}}$ -level equivalence and hence so is the induced map from the first term to the mapping telescope $\tilde{\alpha}_Z^{\mathcal{U}} : Z \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}} Z$. \square

Proposition 3.45. *If X is $G^{\mathcal{U}}$ -semistable, then $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is a $G^{\mathcal{U}}\Omega$ -spectrum.*

Proof. Let H be a subgroup of G and M, N two finite disjoint H -subsets of \mathcal{U} . Since X is $G^{\mathcal{U}}$ -semistable, so is $sh^M X$ by Corollary 3.42 and hence the map $\tilde{\alpha}_{sh^M X}^N$ is a $\pi_*^{\mathcal{U}}$ -isomorphism. Now Lemma 3.44 implies that in this case the adjoint structure map $\tilde{\sigma}_M^N : (\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M) \rightarrow \Omega^N(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M \sqcup N)$ induces a bijection on all homotopy groups at the basepoint of the H -fixed points.

We are not yet done because we have not shown anything about the homotopy groups at elements of $(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M)^H$ which lie in other components. For this we use that there is a

group-like H -space structure on $(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M)^H$ and $(\Omega^N(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)(M \sqcup N))^H$ for which $\tilde{\sigma}_M^N$ is a homomorphism, obtained in the following way: We take some $n \in \mathbb{N}$ such that M_n is not the empty set. Then $(\Omega^{\mathcal{U}} sh^{\mathcal{U}} X)$ is G -homotopy equivalent to the mapping telescope only taken over the terms $\Omega^{M_i} sh^{M_i} X$ for $i \geq n$. From n on all the maps in the system are of the form $\Omega^{M_i} f_i$ for some map f_i , in particular all the maps are loop maps with respect to M_n . Since taking a loop space commutes with mapping telescopes up to weak equivalence, we obtain that $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is $G^{\mathcal{U}}$ -level equivalent to Ω of some spectrum, so since $\mathbb{R}[M_n]$ contains a trivial summand, the levels obtain a group-like H -space structure. Because the adjoint structure maps of the loops of a spectrum are given by the loops of the adjoint structure maps, we see that $\tilde{\sigma}_M^N$ is a homomorphism with respect to these H -space structures and so it is a genuine H -equivalence. \square

Together with Lemma 3.30 and Corollary 3.43 this gives:

Corollary 3.46. *A G -symmetric spectrum allows a $\pi_*^{\mathcal{U}}$ -isomorphism to a $G^{\mathcal{U}}\Omega$ -spectrum if and only if it is $G^{\mathcal{U}}$ -semistable.*

For convenience we collect the various results on equivariant semistability in one statement:

Theorem 3.47 (Semistability). *The naive equivariant homotopy groups of a G -symmetric spectrum carry a natural additive action of the monoid of G -equivariant injective self-maps of the G -set universe \mathcal{U} , compatible with the $G^{\mathcal{U}}$ -Mackey functor structure. A G -symmetric spectrum allows a $\pi_*^{\mathcal{U}}$ -isomorphism to a $G^{\mathcal{U}}\Omega$ -spectrum if and only if this action is trivial. Furthermore, the class for which this is the case is closed under wedges, shifts along finite G -subsets of \mathcal{U} , cones, smash products (as long as one factor is also G -flat), $\text{map}(A, -)$ for a finite based G -CW complex A , restriction to a subgroup and both induction and coinduction from a subgroup. It contains all restrictions of G -orthogonal spectra (in particular, suspension spectra) and all G -symmetric spectra X for which every naive homotopy group $\pi_n^{H, \mathcal{U}} X$ is finitely generated.*

Proof. The action is constructed in Section 3.4 and Corollary 3.46 proves that the triviality of this action is equivalent to the existence of a $\pi_*^{\mathcal{U}}$ -isomorphism to a $G^{\mathcal{U}}\Omega$ -spectrum. The class of such G -symmetric spectra is preserved under wedges as a consequence of Corollary 3.10, under shifts by Corollary 3.42, under cones by applying the five lemma to the operators d^M and the long exact sequences of Proposition 3.8, under smash products with one G -flat factor by Proposition 7.6, under $\text{map}(A, -)$ for finite A by Corollary 3.41, under induction and coinduction from a subgroup by Lemma 3.13 and the preceding discussion and under restriction to a subgroup by definition. It contains all G -orthogonal spectra, because these allow a $\pi_*^{\mathcal{U}}$ -isomorphism to a $G^{\mathcal{U}}\Omega$ -spectrum and all G -symmetric spectra with finitely generated naive homotopy groups by Corollary 3.31. \square

3.7 Relation to $G^{\mathcal{U}}$ -stable equivalences

We are now ready for:

Theorem 3.48. *Every $\pi_*^{\mathcal{U}}$ -isomorphism of G -symmetric spectra is a $G^{\mathcal{U}}$ -stable equivalence.*

Proof. The proof is an equivariant analog of the argument in [HSS00, Theorem 3.1.11]. We first show the following special case: If a G -symmetric spectrum of spaces has trivial homotopy groups, it is $G^{\mathcal{U}}$ -stably contractible. So we let X be such a G -symmetric spectrum. By Corollary 3.36 we know that all shifts $sh^M X$ along finite G -subsets M of \mathcal{U} also have trivial homotopy groups and so by Lemma 3.44 the spectrum $\Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ is $G^{\mathcal{U}}$ -level contractible and in particular $G^{\mathcal{U}}$ -stably contractible.

We claim that this implies that X is $G^\mathcal{U}$ -stably contractible, too: The functors $\Omega^{M_i} sh^{M_i}$ preserve $G^\mathcal{U}$ -level equivalences and hence so does their mapping telescope $\Omega^\mathcal{U} sh^\mathcal{U}$. Therefore, we obtain a functor

$$\Omega^\mathcal{U} sh^\mathcal{U} : \mathrm{Ho}_{lev}^\mathcal{U}(GSp_\mathbf{T}^\Sigma) \rightarrow \mathrm{Ho}_{lev}^\mathcal{U}(GSp_\mathbf{T}^\Sigma)$$

and a natural transformation $\tilde{\alpha} : id_{\mathrm{Ho}_{lev}^\mathcal{U}(GSp_\mathbf{T}^\Sigma)} \rightarrow \Omega^\mathcal{U} sh^\mathcal{U}$. Let $\gamma : GSp_\mathbf{T}^\Sigma \rightarrow \mathrm{Ho}_{lev}^\mathcal{U}(GSp_\mathbf{T}^\Sigma)$ be the projection. In the following we abbreviate $\mathrm{Ho}_{lev}^\mathcal{U}(GSp_\mathbf{T}^\Sigma)$ by $\mathrm{Ho}_{lev}^\mathcal{U}$. We have to show that $\mathrm{Ho}_{lev}^\mathcal{U}(X, Z)$ is trivial for all $G^\mathcal{U}\Omega$ -spectra Z . We consider the composite

$$\mathrm{Ho}_{lev}^\mathcal{U}(X, Z) \xrightarrow{\Omega^\mathcal{U} sh^\mathcal{U}} \mathrm{Ho}_{lev}^\mathcal{U}(\Omega^\mathcal{U} sh^\mathcal{U} X, \Omega^\mathcal{U} sh^\mathcal{U} Z) \xrightarrow{\tilde{\alpha}_X^*} \mathrm{Ho}_{lev}^\mathcal{U}(X, \Omega^\mathcal{U} sh^\mathcal{U} Z).$$

It sends a morphism $\phi : X \rightarrow Z$ to $\Omega^\mathcal{U} sh^\mathcal{U}(\phi) \circ \tilde{\alpha}_X$, which by naturality equals $\tilde{\alpha}_Z \circ \phi$. Since Z is a $G^\mathcal{U}\Omega$ -spectrum, it follows from Lemma 3.44 that $\tilde{\alpha}_Z$ is an isomorphism in the level homotopy category and hence postcomposition with it gives a natural isomorphism on morphism sets. It follows that $\mathrm{Ho}_{lev}^\mathcal{U}(X, Z)$ is a retract of $\mathrm{Ho}_{lev}^\mathcal{U}(\Omega^\mathcal{U} sh^\mathcal{U} X, Z)$ which we know to be trivial since $\Omega^\mathcal{U} sh^\mathcal{U} X$ is $G^\mathcal{U}$ -stably contractible. Hence, $\mathrm{Ho}_{lev}^\mathcal{U}(X, Z)$ is trivial, too, and X is $G^\mathcal{U}$ -stably contractible.

As a next step we show that any $\pi_*^\mathcal{U}$ -isomorphism between G -symmetric spectra of spaces is a $G^\mathcal{U}$ -stable equivalence. So let $f : X \rightarrow Y$ be a $\pi_*^\mathcal{U}$ -isomorphism. Since every $G^\mathcal{U}$ -level equivalence is both a $\pi_*^\mathcal{U}$ -isomorphism and a $G^\mathcal{U}$ -stable equivalence, we can assume that X and Y are G -flat and in addition that f is a G -flat cofibration. Then the cone $C(f)$ of f is G -flat, too. By the long exact sequence in Proposition 3.8, we furthermore know that $C(f)$ has trivial homotopy groups. By the case already considered, it is $G^\mathcal{U}$ -stably contractible. We now have to show that f induces a bijection on G -homotopy classes into $G^\mathcal{U}$ -flat level-fibrant $G^\mathcal{U}\Omega$ -spectra Z . Since for any such Z and any finite G -subset M of \mathcal{U} the map $\tilde{\alpha}_Z^M : Z \rightarrow \Omega^M sh^M Z$ is a $G^\mathcal{U}$ -level equivalence (cf. Example 2.41), it suffices to show that f induces bijections on G -homotopy classes into $\Omega^M sh^M Z$, where we pick an M which contains at least two G -orbits. There is an exact sequence

$$[C(f), \Omega^M sh^M Z]^G \xrightarrow{i(f)^*} [Y, \Omega^M sh^M Z]^G \xrightarrow{f^*} [X, \Omega^M sh^M Z]^G \rightarrow [C(f), \Omega^{M-\mathbb{R}} sh^M Z]^G.$$

In the last step we have adjoined a sphere to the left to be able to continue the sequence. The terms on the left and on the right are trivial, since $C(f)$ is G -flat and $G^\mathcal{U}$ -stably contractible and the targets are $G^\mathcal{U}\Omega$ -spectra. Furthermore, this is an exact sequence of groups (using the remaining trivial loop coordinate in $\Omega^{M-\mathbb{R}}$) and hence the induced map in the middle is a bijection. Thus, f is a $G^\mathcal{U}$ -stable equivalence.

The statement for G -symmetric spectra of simplicial sets follows from the one over spaces: By definition, if a map $f : X \rightarrow Y$ of G -symmetric spectra of simplicial sets is a $\pi_*^\mathcal{U}$ -isomorphism, then so is its geometric realization $|f|$. Hence, $|f|$ is a $G^\mathcal{U}$ -stable equivalence, which is equivalent to f being one by Proposition 2.47. \square

Corollary 3.49. *A map between $G^\mathcal{U}$ -semistable G -symmetric spectra is a $\pi_*^\mathcal{U}$ -isomorphism if and only if it is a $G^\mathcal{U}$ -stable equivalence.*

Proof. It suffices to show the topological case. We have just seen that every $\pi_*^\mathcal{U}$ -isomorphism is a $G^\mathcal{U}$ -stable equivalence. For the other direction, let $f : X \rightarrow Y$ be a $G^\mathcal{U}$ -stable equivalence between $G^\mathcal{U}$ -semistable X and Y . Then $\Omega^\mathcal{U} sh^\mathcal{U}(f)$ is a $G^\mathcal{U}$ -stable equivalence between $G^\mathcal{U}\Omega$ -spectra, hence a $G^\mathcal{U}$ -level equivalence by the Yoneda Lemma and in particular a $\pi_*^\mathcal{U}$ -isomorphism. By 2-out-of-3 it follows that f is also a $\pi_*^\mathcal{U}$ -isomorphism. \square

4 Stable model structures

4.1 Properties of $G^{\mathcal{U}}$ -stable equivalences

This section examines the behavior of $G^{\mathcal{U}}$ -stable equivalences with respect to certain constructions.

Proposition 4.1. (i) *A wedge of $G^{\mathcal{U}}$ -stable equivalences is again a $G^{\mathcal{U}}$ -stable equivalence.*
(ii) *Smashing with a cofibrant G -space A preserves $G^{\mathcal{U}}$ -stable equivalences.*

Proof. Regarding (i): Let $(f_i : X_i \rightarrow Y_i)_{i \in I}$ be a collection of $G^{\mathcal{U}}$ -stable equivalences. If all the X_i and Y_i are G -flat one can make use of the universal property of the wedge to show that $\bigvee f_i$ also induces bijections on G -homotopy classes into $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ spectra. In general one can reduce to this case because by Corollary 3.10 the map $\bigvee X_i^b \rightarrow \bigvee X_i$ is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism and hence a $G^{\mathcal{U}}$ -stable equivalence.

The second statement follows by adjunction in the case where domain and target are G -flat, since smashing with A preserves G -flat symmetric spectra and $\text{map}(A, -)$ preserves $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectra. Corollary 3.14 allows to reduce to this case. \square

In fact, one can use Corollary 3.14 and induct over the cells of A to show that if A has isotropy in a family \mathcal{F} of subgroups of G , then smashing with A takes every map which is an $H^{\mathcal{U}}$ -stable equivalence for all $H \in \mathcal{F}$ to a $G^{\mathcal{U}}$ -stable equivalence.

Proposition 4.2. *Let $f : X \rightarrow Y$ be a map of G -symmetric spectra of spaces and V a G -subrepresentation of $\mathbb{R}[\mathcal{U}]$. Then the following are equivalent:*

- (i) *f is a $G^{\mathcal{U}}$ -stable equivalence.*
- (ii) *$S^V \wedge f$ is a $G^{\mathcal{U}}$ -stable equivalence.*
- (iii) *$\Omega^V f$ is a $G^{\mathcal{U}}$ -stable equivalence.*
- (iv) *$C(f)$ is $G^{\mathcal{U}}$ -stably contractible, i.e., the unique map $C(f) \rightarrow *$ is a $G^{\mathcal{U}}$ -stable equivalence.*
- (v) *$H(f)$ is $G^{\mathcal{U}}$ -stably contractible.*

Proof. The implication (i) \Rightarrow (ii) follows from Proposition 4.1.

(ii) \Rightarrow (i): We assume $S^V \wedge f$ to be a $G^{\mathcal{U}}$ -stable equivalence and let Z be a level fibrant $G^{\mathcal{U}}\Omega$ -spectrum. We further pick a G -flat replacement $f^b : X^b \rightarrow Y^b$ of f and an embedding $V \rightarrow \mathbb{R}[M]$ for some finite G -subset M of \mathcal{U} . By Corollary 2.42, the natural map $Z \rightarrow \Omega^M sh^M Z$ is a $G^{\mathcal{U}}$ -level equivalence and thus there are isomorphisms of maps:

$$[f^b, Z]^G \cong [f^b, \Omega^M sh^M Z]^G \cong [S^M \wedge f^b, sh^M Z]^G \cong [S^{M-V} \wedge S^V \wedge f^b, sh^M Z]^G$$

Now, via the same argument as for the first implication, the G -map $S^M \wedge f^b$ is a G -stable equivalence, too. Since smashing with S^{M-V} preserves $G^{\mathcal{U}}$ -stable equivalences and the G -spectrum $sh^M Z$ is again a level-fibrant $G^{\mathcal{U}}\Omega$ -spectrum (Example 2.41), the latter map is a bijection.

(i) \Leftrightarrow (iii): The map $\Omega^V f$ is a $G^{\mathcal{U}}$ -stable equivalence if and only if $S^V \wedge \Omega^V f$ is by (ii), which is equivalent to f being one, since the counit is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism and hence $G^{\mathcal{U}}$ -stable equivalence.

(i) \Leftrightarrow (iv): We first assume that X and Y are G -flat and that f is a G -flat cofibration. Then the cone $C(f)$ is G -flat, too. Mapping out of the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f)$ yields a long exact sequence

$$\dots \rightarrow [S^1 \wedge Y, Z]^G \xrightarrow{(S^1 \wedge f)^*} [S^1 \wedge X, Z]^G \xrightarrow{q(f)^*} [C(f), Z]^G \xrightarrow{i(f)^*} [Y, Z]^G \xrightarrow{f^*} [X, Z]^G.$$

This sequence is in fact naturally one of abelian groups, since Z is G -level equivalent to $\Omega^2 sh^2 Z$. In view of the equivalence of (i) and (ii), we see that:

$$\begin{aligned} f \text{ is a } G^{\mathcal{U}}\text{-stable equivalence} &\Leftrightarrow S^n \wedge f \text{ is a } G^{\mathcal{U}}\text{-stable equivalence for all } n \geq 0 \\ &\Leftrightarrow S^n \wedge C(f) \text{ is } G^{\mathcal{U}}\text{-stably contractible for all } n \geq 0 \\ &\Leftrightarrow C(f) \text{ is } G^{\mathcal{U}}\text{-stably contractible} \end{aligned}$$

In general, we can first replace $f : X \rightarrow Y$ by $f^b : X^b \rightarrow Y^b$ and then factor f^b as a cofibration followed by a $G^{\mathcal{U}}$ -level equivalence. In either step the cone only changes up to $\pi_*^{\mathcal{U}}$ -isomorphism by the five lemma applied to the natural long exact sequence associated to the cone in Proposition 3.8. Furthermore, f is a $G^{\mathcal{U}}$ -stable equivalence if and only if this resulting map is. Hence, the result follows from the special case above.

(iv) \Leftrightarrow (v): From Corollary 3.9 we know that the natural map from $S^1 \wedge H(f)$ to $C(f)$ is a $\pi_*^{\mathcal{U}}$ -isomorphism. Using the equivalence of (i) and (ii), we obtain that under these circumstances $H(f)$ is $G^{\mathcal{U}}$ -stably contractible if and only if $C(f)$ is. \square

The proposition also holds for maps between G -symmetric spectra of simplicial sets (only allowing finite G -sets M instead of V or defining $S^V = \mathcal{S}(S^V)$), but for items (iii) and (v) one needs the condition that X and Y are $G^{\mathcal{U}}$ -projective level fibrant.

For the following we first need the following definition:

Definition 4.3 (*h-cofibrations*). A map $i : X \rightarrow Y$ of G -symmetric spectra of spaces is called an *h-cofibration* if it satisfies the homotopy extension property, i.e., if for every map $f : Y \rightarrow Z$ and every homotopy $H : [0, 1]_+ \wedge X \rightarrow Z$ such that $H|_{\{0\}_+ \wedge X} = f \circ i$, there exists a homotopy $\tilde{H} : [0, 1]_+ \wedge Y \rightarrow Z$ with $\tilde{H} \circ ([0, 1]_+ \wedge i) = H$ and $\tilde{H}|_{\{0\}_+ \wedge Y} = f$.

This property is equivalent to the inclusion $([0, 1]_+ \wedge X) \cup_{\{0\}_+ \wedge X} Y \rightarrow [0, 1]_+ \wedge Y$ having a retraction, and so it follows that any functor preserving pushouts and the smash product with the interval also preserves *h-cofibrations*. In particular, this holds for smashing with any G -symmetric spectrum. Furthermore, the pushout of an *h-cofibration* along an arbitrary map is again an *h-cofibration*, and so is the (transfinite) composition of *h-cofibrations*. Since it is easily checked that the generating G -flat cofibrations (Equation 2.1) are *h-cofibrations*, this implies that any G -flat cofibration (and hence also any $G^{\mathcal{U}}$ -projective cofibration) is an *h-cofibration*.

Proposition 4.4. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{g} & Y \end{array}$$

be a pushout diagram of G -symmetric spectra of spaces with i an h -cofibration. Then we have:

- (i) *If i is a $G^{\mathcal{U}}$ -stable equivalence, then so is j .*
- (ii) *If f is a $G^{\mathcal{U}}$ -stable equivalence, then so is g .*

In fact the condition that i is an *h-cofibration* can be weakened to every level $i(M)$ for $M \subseteq \mathcal{U}$ being an *h-cofibration* of based G -spaces. In particular, geometrically realizing a map of G -symmetric spectra of simplicial sets which is levelwise injective gives such a map. There is also a dual statement in the case where the diagram is a pullback and j a $G^{\mathcal{U}}$ -projective level fibration. Moreover, analogous statements for G -symmetric spectra of simplicial sets can be obtained by applying the topological one to their geometric realization.

Proof. (i): If i is a $G^{\mathcal{U}}$ -stable equivalence, then its cone is $G^{\mathcal{U}}$ -stably contractible by Proposition 4.2. Since i is also an h -cofibration, the cone is $G^{\mathcal{U}}$ -level equivalent to the quotient X/A , which in turn is isomorphic to the quotient Y/B . Hence, the latter is also $G^{\mathcal{U}}$ -stably contractible and so is the cone $C(j)$, since j is an h -cofibration. Applying Proposition 4.2 once more gives that j is a $G^{\mathcal{U}}$ -stable equivalence.

(ii): We know that the cone of f is $G^{\mathcal{U}}$ -stably contractible and hence it suffices to show that the map $C(f) \rightarrow C(g)$ is a $G^{\mathcal{U}}$ -stable equivalence. This in turn can be checked by looking at its mapping cone, which is isomorphic to the mapping cone of the map $C(i) \rightarrow C(j)$. But since i and j are h -cofibrations, these cones are $G^{\mathcal{U}}$ -level equivalent to the quotients, between which the induced map is an isomorphism. This finishes the proof. \square

Proposition 4.5. *Let $X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \dots$ be a sequence of morphisms i_j of G -symmetric spectra of spaces which are h -cofibrations and $G^{\mathcal{U}}$ -stable equivalences. Then the induced morphism $X_0 \rightarrow \operatorname{colim}_j X_j$ is also a $G^{\mathcal{U}}$ -stable equivalence.*

Proof. By inductively replacing the maps if necessary, we can assume that X_0 is G -flat and that all i_j are G -flat cofibrations. Hence, all the X_j and the colimit $\operatorname{colim}_j X_j$ are G -flat, too. Now let Z be a $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}$ - Ω -spectrum. Then there is a G -homeomorphism of spaces

$$\operatorname{map}_{Sp^{\Sigma}}(\operatorname{colim}_j X_j, Z) \cong \lim_j \operatorname{map}_{Sp^{\Sigma}}(X_j, Z).$$

Since the level model structure is G -topological (Proposition 2.38), all the maps $\operatorname{map}_{Sp^{\Sigma}}(i_j, Z) : \operatorname{map}_{Sp^{\Sigma}}(X_{j+1}, Z) \rightarrow \operatorname{map}_{Sp^{\Sigma}}(X_j, Z)$ are genuine G -fibrations. Hence, the statement follows from the fact that if in such a system of genuine G -fibrations each map induces a bijection on path components of the G -fixed points, then so does the induced map from the limit to the first term. \square

As above, in the simplicial version it suffices that all maps i_j are levelwise injective.

4.2 $Q^{\mathcal{U}}$ -replacement and stable model structures

The next goal is the construction of an endofunctor $Q^{\mathcal{U}}$ of G -symmetric spectra with image in $G^{\mathcal{U}}$ - Ω -spectra, together with a natural $G^{\mathcal{U}}$ -stable equivalence $q^{\mathcal{U}} : id \rightarrow Q^{\mathcal{U}}$. We note that for $G^{\mathcal{U}}$ -semistable G -symmetric spectra the construction $\tilde{\alpha}_X^{\mathcal{U}} : X \rightarrow \Omega^{\mathcal{U}} sh^{\mathcal{U}} X$ of Section 3.6 has these properties, but for general X one needs to proceed in a different way.

We recall the $G^{\mathcal{U}}$ -stable equivalences

$$G \ltimes_H \lambda_{M,N}^{(H)} : G \ltimes_H \mathcal{F}_{M \sqcup N}^{(H)} S^N \rightarrow G \ltimes_H \mathcal{F}_M^{(H)} S^0$$

from Example 2.46. In general they are not G -flat cofibrations, in fact not even levelwise injective, as the case $G = H = \{e\}$, $M = \emptyset$ and $N = 1$ shows. To correct this we recall that the mapping cylinder $\operatorname{Cyl}(f)$ of a map of G -symmetric spectra $f : X \rightarrow Y$ is defined as the pushout of f along the inclusion $X \xrightarrow{i_1} [0, 1]_+ \wedge X$ (with Δ^1 taking the place of the interval in the simplicial case). We use it to factor $G \ltimes_H \lambda_{M,N}^{(H)}$ as

$$G \ltimes_H \mathcal{F}_{M \sqcup N}^{(H)} S^N \xrightarrow{G \ltimes_H \tilde{\lambda}_{M,N}^{(H)}} G \ltimes_H \operatorname{Cyl}(\lambda_{M,N}^{(H)}) \xrightarrow{G \ltimes_H r_{M,N}^{(H)}} G \ltimes_H \mathcal{F}_M^{(H)} S^0.$$

Here, the map $G \ltimes_H r_{M,N}^{(H)}$ is a G -homotopy equivalence and $\tilde{\lambda}_{M,N}^{(H)}$ is a $G^{\mathcal{U}}$ -projective cofibration. The latter is a formal consequence of the fact that $G \ltimes_H \mathcal{F}_{M \sqcup N}^{(H)} S^N$ and $G \ltimes_H \mathcal{F}_M^{(H)} S^0$ are $G^{\mathcal{U}}$ -projective as well as the $G^{\mathcal{U}}$ -projective level model structures being topological (or simplicial). For example, this is explained in the proof of [HSS00, Lemma 3.4.10].

We define

$$J_{\mathcal{U},proj}^{st} := \{i\Box(G \ltimes_H \overline{\lambda}_{M,N}^{(H)}) \mid i \in I_{\{e\}}, H \leq G, M, N \subseteq_H \mathcal{U}\} \cup J_{\mathcal{U},proj}^{lev}.$$

Again, \Box denotes the pushout product with respect to the smash product and $I_{\{e\}}$ is a set of generating cofibrations of the Quillen model structures on non-equivariant topological spaces or simplicial sets.

Lemma 4.6. *For a $G^{\mathcal{U}}$ -projectively level fibrant G -symmetric spectrum X , H a subgroup of G and M and N two finite H -subsets of \mathcal{U} the following are equivalent:*

1. *The map $(\tilde{\sigma}_M^N)^H : X(M)^H \rightarrow \text{map}_H(S^N, X(M \sqcup N))$ is a weak homotopy equivalence.*
2. *X has the right lifting property with respect to the set $\{i\Box(G \ltimes_H \overline{\lambda}_{M,N}^{(H)})\}_{i \in I_{\{e\}}}$.*

Proof. By adjunction, X has the right lifting property with respect to $\{i\Box(G \ltimes_H \overline{\lambda}_{M,N}^{(H)})\}_{i \in I_{\{e\}}}$ if and only if

$$\text{map}_{GSp^{\Sigma}}(G \ltimes_H \overline{\lambda}_{M,N}^{(H)}, X) : \text{map}_{GSp^{\Sigma}}(G \ltimes_H \text{Cyl}(\lambda_{M,N}^{(H)}), X) \rightarrow \text{map}_{GSp^{\Sigma}}(G \ltimes_H \mathcal{F}_{M \sqcup N}^{(H)} S^N, X)$$

has the right lifting property with respect to the set $I_{\{e\}}$. Since, by Proposition 2.38, the level model structure is topological (or simplicial), this map is always a Serre-(or Kan-)fibration. Hence, it has the right lifting property with respect to $I_{\{e\}}$ if and only if it is a weak homotopy equivalence. Since $G \ltimes_H r_{M,N}^{(H)}$ is a G -homotopy equivalence, this in turn is equivalent to

$$\text{map}_{GSp^{\Sigma}}(G \ltimes_H \mathcal{F}_M S^0, X) \xrightarrow{\text{map}_{GSp^{\Sigma}}(G \ltimes_H \lambda_{M,N}^{(H)}, X)} \text{map}_{GSp^{\Sigma}}(G \ltimes_H \mathcal{F}_{M \sqcup N} S^N, X)$$

being a weak homotopy equivalence. Via two adjunctions this map is isomorphic to

$$X(M)^H \xrightarrow{(\tilde{\sigma}_M^N)^H} \text{map}_H(S^N, X(M \sqcup N)),$$

so we are done. \square

Corollary 4.7. *A G -symmetric spectrum of spaces or simplicial sets is a $G^{\mathcal{U}}\Omega$ -spectrum if and only if it has the right lifting property with respect to the set $J_{\mathcal{U},proj}^{st}$.*

Proof. Since we already know (cf. Section 2.9) that a G -symmetric spectrum is $G^{\mathcal{U}}$ -projectively level fibrant if and only if it has the right lifting property with respect to the set $J_{\mathcal{U},proj}^{lev}$, this is a direct consequence of the previous lemma. \square

It follows from Proposition 2.38, i.e., the fact that the level model structures are G -topological (or G -simplicial), that every map in $J_{\mathcal{U},proj}^{st}$ is a $G^{\mathcal{U}}$ -projective cofibration. Also, all domains (and codomains) of maps in $J_{\mathcal{U},proj}^{st}$ are small with respect to countably infinite sequences of flat cofibrations. Hence we can apply the small object argument (cf. [Hir03, 10.5.16]) to obtain a functor $Q^{\mathcal{U}} : GSp^{\Sigma} \rightarrow GSp^{\Sigma}$ with image in $G^{\mathcal{U}}\Omega$ -spectra and a natural transformation $q^{\mathcal{U}} : id \rightarrow Q^{\mathcal{U}}$.

We are left to show that for every G -symmetric spectrum X the map $q_X^{\mathcal{U}} : X \rightarrow Q^{\mathcal{U}}X$ is a $G^{\mathcal{U}}$ -stable equivalence. We first prove the following:

Lemma 4.8. *Let $i : A \rightarrow B$ be a genuine G -cofibration of based G -spaces (or based G -simplicial sets) and $f : X \rightarrow Y$ a G -flat cofibration of G -symmetric spectra which is also a $G^{\mathcal{U}}$ -stable equivalence. Then the pushout product map*

$$i\Box f : (B \wedge X) \cup_{A \wedge X} (A \wedge Y) \rightarrow B \wedge Y$$

is again a G -flat cofibration and a $G^{\mathcal{U}}$ -stable equivalence.

Proof. We already know that $i\Box f$ is again a G -flat cofibration, since the level model structure is based G -topological (or based G -simplicial) by Proposition 2.38. Hence, using Proposition 4.2 and the fact that for G -flat cofibrations the cone is $G^{\mathcal{U}}$ -level equivalent to the cofiber, we can prove that $i\Box f$ is a $G^{\mathcal{U}}$ -stable equivalence by showing that its cofiber $B \wedge Y / ((B \wedge X) \cup_{A \wedge X} (A \wedge Y))$ is $G^{\mathcal{U}}$ -stably contractible. This cofiber is G -isomorphic to $(B/A) \wedge (Y/X)$. Since f is a G -flat cofibration and a $G^{\mathcal{U}}$ -stable equivalence, the G -symmetric spectrum Y/X is $G^{\mathcal{U}}$ -stably contractible. Hence, so is its smash product with the genuinely G -cofibrant based space (or based G -simplicial set) B/A (by Proposition 4.1). So $i\Box f$ is a $G^{\mathcal{U}}$ -stable equivalence. \square

Using this we obtain:

Proposition 4.9. *The map $q_X^{\mathcal{U}} : X \rightarrow Q^{\mathcal{U}}X$ is a $G^{\mathcal{U}}$ -stable equivalence for every G -symmetric spectrum X .*

Proof. The map q_X is a relative $J_{\mathcal{U},proj}^{st}$ -complex and hence, by the results of Section 4.1, it suffices to show that every map in $J_{\mathcal{U},proj}^{st}$ is a $G^{\mathcal{U}}$ -stable equivalence and an h -cofibration. Every map in J_G^{lev} is even a $G^{\mathcal{U}}$ -level equivalence and a $G^{\mathcal{U}}$ -projective cofibration, since they are generators for the $G^{\mathcal{U}}$ -level model structure. All the maps $G \rtimes_H \overline{\lambda}_{M,N}^{(H)}$ are $G^{\mathcal{U}}$ -stable equivalences by Example 2.46 and $G^{\mathcal{U}}$ -projective cofibrations. Hence, the statement follows from Lemma 4.8, as every $G^{\mathcal{U}}$ -projective cofibration is in particular a G -flat cofibration.. \square

We are now ready to establish the $G^{\mathcal{U}}$ -stable model structures for G -symmetric spectra of spaces and of simplicial sets. We call a map a (positive) *flat $G^{\mathcal{U}}$ -stable fibration* if it has the right lifting property with respect to all maps that are (positive) G -flat cofibrations and $G^{\mathcal{U}}$ -stable equivalences. Similarly, it is called a (positive) *projective $G^{\mathcal{U}}$ -stable fibration* if it has the right lifting property with respect to all maps that are (positive) projective $G^{\mathcal{U}}$ -cofibrations and $G^{\mathcal{U}}$ -stable equivalences. Then we have:

Theorem 4.10 (Flat stable model structures). *The classes of (positive) G -flat cofibrations, $G^{\mathcal{U}}$ -stable equivalences and (positive) $G^{\mathcal{U}}$ -stable fibrations define a cofibrantly generated, proper model structure on the categories of G -symmetric spectra of spaces and simplicial sets, called the (positive) $G^{\mathcal{U}}$ -flat stable model structure.*

Theorem 4.11 (Projective stable model structures). *The classes of (positive) projective $G^{\mathcal{U}}$ -cofibrations, $G^{\mathcal{U}}$ -stable equivalences and (positive) projective $G^{\mathcal{U}}$ -stable fibrations define a cofibrantly generated, proper model structure on the categories of G -symmetric spectra of spaces and simplicial sets, called the (positive) $G^{\mathcal{U}}$ -projective stable model structure.*

Proof. All the model structures are obtained in the same way by applying left Bousfield localization to the respective level model structure,. We apply Theorem [Bou01, Theorem 9.3], with respect to the functor $Q^{\mathcal{U}}$ and the natural transformation $q^{\mathcal{U}} : id \rightarrow Q^{\mathcal{U}}$ just constructed. Any $G^{\mathcal{U}}$ -stable equivalence between $G^{\mathcal{U}}\Omega$ -spectra is a $G^{\mathcal{U}}$ -level equivalence by the Yoneda Lemma and thus a map f is a $G^{\mathcal{U}}$ -stable equivalence if and only if $Q^{\mathcal{U}}f$ is a $G^{\mathcal{U}}$ -level equivalence (and if and only if $Q^{\mathcal{U}}f$ is a positive $G^{\mathcal{U}}$ -level equivalence). In other words the class of $G^{\mathcal{U}}$ -stable equivalences agrees with that of $Q^{\mathcal{U}}$ -equivalences in the sense of Bousfield's theorem.

We now check the axioms in [Bou01]. (A1) requires that every $G^{\mathcal{U}}$ -level equivalence is a $G^{\mathcal{U}}$ -stable equivalence, which is clear. For every G -symmetric spectrum X the maps $q_{QX}^{\mathcal{U}}$ and $Q^{\mathcal{U}}q_X^{\mathcal{U}}$ from $Q^{\mathcal{U}}X$ to $Q^{\mathcal{U}}Q^{\mathcal{U}}X$ are $G^{\mathcal{U}}$ -stable equivalences between $G^{\mathcal{U}}\Omega$ -spectra and hence $G^{\mathcal{U}}$ -level equivalences, so (A2) is satisfied. In (A3) we are given a pullback square

$$\begin{array}{ccc} V & \xrightarrow{k} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{h} & Y \end{array}$$

where X and Y are level-fibrant (positive) $G^\mathcal{U}\Omega$ -spectra, f is a $G^\mathcal{U}$ -level fibration and h a $G^\mathcal{U}$ -stable equivalence. We have to show that then k is a $G^\mathcal{U}$ -stable equivalence, too. This is a direct application of the dual version of Proposition 4.4 (and does not need the hypothesis that X and Y are (positive) $G^\mathcal{U}\Omega$ -spectra). \square

Using the characterization given in [Bou01, Theorem 9.3], a map $f : X \rightarrow Y$ is a $G^\mathcal{U}$ -flat (resp. $G^\mathcal{U}$ -projective) stable fibration if and only if it is a $G^\mathcal{U}$ -flat (resp. $G^\mathcal{U}$ -projective) level fibration and the square

$$\begin{array}{ccc} X & \xrightarrow{q_X^\mathcal{U}} & Q^\mathcal{U}X \\ f \downarrow & & \downarrow Q^\mathcal{U}f \\ Y & \xrightarrow{q_Y^\mathcal{U}} & Q^\mathcal{U}Y \end{array}$$

is homotopy cartesian in the respective level model structure. This shows that generating acyclic cofibrations in all of these model structures can be obtained by adding the maps $i \square_H G \times_H \bar{\lambda}_{M,N}$ for all H -subsets M and N of \mathcal{U} (with $M \neq \emptyset$ in the positive case) to the respective set of generating acyclic cofibrations for the level model structure (while nothing needs to be added to the generating cofibrations). Moreover, a G -symmetric spectrum is fibrant in either of the (positive) stable model structures if and only if it is a level-fibrant (positive) $G^\mathcal{U}\Omega$ -spectrum.

We want to note the following convenient properties of cofibrations of G -symmetric spectra of simplicial sets:

Remark 4.12. Let $f : X \rightarrow Y$ be a map of G -symmetric spectra of simplicial sets. Then the following hold:

- (i) f is a (positive) G -flat cofibration if and only if it is a non-equivariant (positive) flat cofibration.
- (ii) If \mathcal{U} is a complete G -set universe (i.e., if every finite G -set can be embedded into it), then f is a $G^\mathcal{U}$ -projective cofibration if and only if it is a non-equivariant projective cofibration.

Proposition 4.13. *The (positive and non-positive) flat and projective $G^\mathcal{U}$ -stable model structures on G -symmetric spectra are monoidal with respect to the smash product.*

Proof. Since the stable model structure shares the cofibrations with the respective level model structure and every $G^\mathcal{U}$ -stably acyclic projective cofibration is also a $G^\mathcal{U}$ -stably acyclic flat cofibration, we only have to show that the pushout product $f \square g$ of a $G^\mathcal{U}$ -stably acyclic flat cofibration $f : A \rightarrow B$ with a flat cofibration $C \rightarrow D$ is again a $G^\mathcal{U}$ -stable equivalence. The proof is the same as for Lemma 4.8, this time making use of the fact that the Hom-spectrum from a flat G -symmetric spectrum into a $G^\mathcal{U}$ -flat level fibrant $G^\mathcal{U}\Omega$ -spectrum is again a flat level fibrant $G^\mathcal{U}\Omega$ -spectrum (which is Corollary 2.43).

To establish monoidality for the positive model structures we furthermore need to show that for a positive $G^\mathcal{U}$ -projective cofibrant replacement $\mathbb{S}^+ \rightarrow \mathbb{S}$ the map $(\mathbb{S}^+ \wedge X) \rightarrow X$ is a $G^\mathcal{U}$ -stable equivalence, but this is clear because every positive $G^\mathcal{U}$ -level equivalence is a $G^\mathcal{U}$ -stable equivalence. \square

Finally, we have the following commutative square of identity Quillen equivalences, where the $+$ denotes the positive model structures:

$$\begin{array}{ccc} GSp_{\mathcal{U},proj,+}^\Sigma & \xrightleftharpoons{\quad} & GSp_{\mathcal{U},proj}^\Sigma \\ \downarrow \uparrow & & \downarrow \uparrow \\ GSp_{\mathcal{U},flat,+}^\Sigma & \xrightleftharpoons{\quad} & GSp_{\mathcal{U},flat}^\Sigma \end{array}$$

Here, the arrows going from the left to the right and from the top to the bottom denote left Quillen functors.

4.3 Some properties of the homotopy category of G -symmetric spectra

We explain certain properties of the homotopy category $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ of the $G^{\mathcal{U}}$ -stable model structures. First we check that they are in fact stable in the sense of model categories, i.e., that the derived suspension functor is an auto-equivalence. As a consequence, the homotopy category $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ has an induced structure of a triangulated category (cf. [Hov99, Chapter 7]). More generally, the homotopy category is stable with respect to all representation spheres S^V for representations V which embed into $\mathbb{R}[\mathcal{U}]$:

Proposition 4.14 (Stability). *Let V be a finite dimensional G -representation which embeds into $\mathbb{R}[\mathcal{U}]$. Then the adjunction*

$$S^V \wedge (-) : GSp_{\mathbb{T}}^{\Sigma} \rightleftarrows GSp_{\mathbb{T}}^{\Sigma} : \Omega^V$$

is a Quillen equivalence for either of the \mathcal{U} -stable model structures of the previous section.

Proof. By Proposition 4.13, the stable model structures are G -topological, therefore the adjunction is a Quillen pair (using Illman's theorem [Ill78] to deduce that S^V admits the structure of a G -CW complex). By Proposition 4.2 we know that both functors preserve and reflect all $G^{\mathcal{U}}$ -stable equivalences and that adjunction unit and counit are $\underline{\pi}_*$ -isomorphisms (Proposition 3.6), thus $G^{\mathcal{U}}$ -stable equivalences by Theorem 3.48. Hence, they form a Quillen equivalence. \square

For $n \in \mathbb{Z}$ we denote by \mathbb{S}^n the n -fold suspension of the sphere spectrum.

Proposition 4.15 (True homotopy groups). *The G -symmetric spectra $G/H_+ \wedge \mathbb{S}^n$ represent the true homotopy groups, i.e., for H a subgroup of G and a $G^{\mathcal{U}}$ -semistable G -symmetric spectrum X there is a natural isomorphism*

$$\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})(G/H_+ \wedge \mathbb{S}^n, X) \cong \pi_n^{H, \mathcal{U}}(X).$$

Proof. Since we have seen that suspension and loop shift the homotopy groups by one in respective directions (and that suspensions of $G^{\mathcal{U}}$ -semistable spectra are $G^{\mathcal{U}}$ -semistable), it suffices to prove the statement for $n = 0$. Every $G^{\mathcal{U}}$ -semistable G -symmetric spectrum allows a $\underline{\pi}_*$ -isomorphism to a $G^{\mathcal{U}}\Omega$ -spectrum, so we can in addition assume that X is a $G^{\mathcal{U}}\Omega$ -spectrum. The statement then follows by adjunction, since

$$\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})(\Sigma^{\infty}G/H_+, X) \cong [G/H_+, X_0]^G = \pi_0(X_0^H) \cong \pi_0^{H, \mathcal{U}}X,$$

where we have used the non-positive $G^{\mathcal{U}}$ -projective stable model structure to compute the morphism set in the homotopy category. The last step uses Example 3.2. \square

It follows that the G -symmetric spectra $G/H_+ \wedge \mathbb{S}^n$ are compact objects (i.e., every map into an infinite direct sum has image in finitely many summands) of the triangulated homotopy category since the homotopy groups of a wedge are given by the direct sum of the homotopy groups of the wedge summands (cf. Corollary 3.10).

We recall that a set \mathbf{X} of compact objects in a triangulated category \mathcal{T} with infinite sums is called a *set of generators* if any object Y of \mathcal{T} which satisfies $\mathcal{T}(\Sigma^n X, Y) = 0$ for all $n \in \mathbb{Z}$ and all $X \in \mathbf{X}$ is a zero object. Since homotopy groups detect $G^{\mathcal{U}}$ -stable equivalences between $G^{\mathcal{U}}$ -semistable G -symmetric spectra and hence in particular $G^{\mathcal{U}}\Omega$ -spectra, we have:

Proposition 4.16 (Generators). *The homotopy category $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ of G -symmetric spectra is generated as a triangulated category by the set of compact objects $\{G/H_+ \wedge \mathbb{S} \mid H \leq G\}$.*

It is a general fact that this implies that the smallest full subcategory of $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ which is closed under suspensions, loops, cones and (infinite) sums and which contains all the $G/H_+ \wedge \mathbb{S}$ is equal to the full $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ (see [SS03, Lemma 2.2.1], for example).

For later reference we note the following:

Remark 4.17 (Direct sums in the homotopy category). The direct sum of a family $(X_i)_{i \in I}$ of G -symmetric spectra is given by their wedge with no need for cofibrantly replacing the X_i first. This is a consequence of Corollary 3.10 and Theorem 3.48. The first implies that a wedge of $G^{\mathcal{U}}$ -level equivalences is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism, which in turn is a $G^{\mathcal{U}}$ -stable equivalence by the second.

5 Comparison to other models

5.1 Quillen equivalence to G -orthogonal spectra

In this section we explain how our model structures relate to the respective ones on G -orthogonal spectra.

We start with the projective case, and there we are cheating a bit, because we really need to compare to a stronger variant of the one of [MM02], where a level equivalence is a map which induces a genuine H -equivalence on all H -subrepresentations of $\mathbb{R}[\mathcal{U}]$, not just on restrictions of G -subrepresentations, and likewise use a stronger notion of $G^{\mathbb{R}[\mathcal{U}]}\Omega$ -spectra. This does not affect the stable equivalences, which are the $\underline{\pi}_*^{\mathbb{R}[\mathcal{U}]}$ -isomorphisms. In the case where $\mathbb{R}[\mathcal{U}]$ is a complete G -representation universe, the positive version of this stronger model structure is used in [HHR14, B.4.].

Theorem 5.1 (Quillen equivalence to G -orthogonal spectra, I). *The adjunction*

$$L : GSp_{\mathbf{T}}^{\Sigma} \rightleftarrows GSp^O : U$$

is a Quillen equivalence for the $G^{\mathcal{U}}$ -projective stable model structure on $GSp_{\mathbf{T}}^{\Sigma}$ and the (strong) $G^{\mathbb{R}[\mathcal{U}]}$ -projective stable model structure on GSp^O .

For the flat model structures, we let \mathcal{V} be a G -representation universe with the property that every finite dimensional G -subrepresentation sits inside a larger one which is isomorphic to the linearization of a G -set. This gives rise to a G -set universe $\mathcal{U}(\mathcal{V})$ by taking an infinite disjoint union of those G -orbits G/H for which $\mathbb{R}[G/H]$ embeds into \mathcal{V} . In particular, we have $\mathbb{R}[\mathcal{U}(\mathcal{V})] \cong \mathcal{V}$.

Theorem 5.2 (Quillen equivalence to G -orthogonal spectra, II). *The adjunction*

$$L : GSp_{\mathbf{T}}^{\Sigma} \rightleftarrows GSp^O : U$$

is a Quillen equivalence for the $G^{\mathcal{U}(\mathcal{V})}$ -flat stable model structure on $GSp_{\mathbf{T}}^{\Sigma}$ and the $G^{\mathcal{V}}$ -flat stable model structure of [Sto11, Section 2.3.3] on GSp^O .

We note that in general a G -set universe \mathcal{U} is not isomorphic to $\mathcal{U}(\mathbb{R}[\mathcal{U}])$, and if it is not, the $G^{\mathcal{U}}$ -flat model structure does not allow a direct comparison to one of the model structures on G -orthogonal spectra. Nevertheless, it is still Quillen equivalent to one via a zig-zag through the projective model structures. For example, taking \mathcal{U} to be a free G -set universe, its linearization is a complete G -representation universe and so $\mathcal{U}(\mathbb{R}[\mathcal{U}])$ is a complete G -set universe.

Proof of Theorem 5.1. In the (strong version of the) $G^{\mathbb{R}[\mathcal{U}]}$ -projective level model structure on GSp^O , a map is a weak equivalence (fibration) if and only if it is a genuine H -equivalence (H -fibration) when evaluated on all finite dimensional H -subrepresentations V of $\mathbb{R}[\mathcal{U}]$ for all

subgroups H of G . Hence, it follows directly from the natural H -homeomorphism $X(\mathbb{R}M) \cong (UX)(M)$ for every subgroup H of G and every finite H -set M (Lemma 2.13) that the forgetful functor U is a right Quillen functor for the level model structures. In particular, the left adjoint L preserves cofibrations. The stable model structure on G -orthogonal spectra is obtained by a left Bousfield localization at the $\pi_*^{\mathbb{R}[U]}$ -isomorphisms, with fibrant objects being exactly the (strong) $G^{\mathbb{R}[U]}\Omega$ -spectra, so the isomorphism of Lemma 2.13 also implies that U preserves fibrant objects. By adjunction, this implies that L maps $G^{\mathcal{U}}$ -projective G -symmetric spectra which are $G^{\mathcal{U}}$ -stably contractible to $G^{\mathbb{R}[U]}$ -projective $G^{\mathbb{R}[U]}$ -stably contractible G -orthogonal spectra. Since both for G -symmetric spectra and for G -orthogonal spectra a projective cofibration is stably acyclic if and only if its cofiber is stably contractible, this implies that L also preserves stable acyclic cofibrations and hence is a left Quillen functor.

It remains to show that the derived functors are equivalences between the homotopy categories. Since a $\pi_*^{\mathcal{U}}$ -isomorphism is a $G^{\mathcal{U}}$ -stable equivalence (Theorem 3.48), we see that U preserves all stable equivalences. In addition, as the underlying G -symmetric spectra of G -orthogonal spectra are all $G^{\mathcal{U}}$ -semistable (they allow $\pi_*^{\mathcal{U}}$ -isomorphisms to $G^{\mathcal{U}}\Omega$ -spectra), Corollary 3.49 implies that U also reflects all stable equivalences. This means that we can use criterion [MMSS01, Lemma A.2.(iii)] and so it suffices to show that the derived unit of the adjunction is a natural isomorphism in the stable \mathcal{U} -homotopy category of G -symmetric spectra. As both model structures are stable, the derived functors $\mathbb{L}L$ and $\mathbb{R}U$ can be equipped with an exact structure (in the sense of triangulated categories) such that unit and counit become exact natural transformations. Both $\mathbb{L}L$ and $\mathbb{R}U$ preserve arbitrary direct sums (which is formal for the left adjoint and follows from Remark 4.17 for $\mathbb{R}U$) and hence so does their composite, so it suffices to check that the derived unit is an isomorphism on the set of compact generators $\{G/H_+ \wedge \mathbb{S} \mid H \leq G\}$ (since the subcategory of objects for which it is an isomorphism is easily shown to be closed under cones, suspensions, loops and arbitrary direct sums). Because each $G/H_+ \wedge \mathbb{S}$ is G -flat and U preserves all $G^{\mathcal{U}}$ -stable equivalences, this in turn is equivalent to checking that for each subgroup H of G the unit $\eta_{G/H_+ \wedge \mathbb{S}} : G/H_+ \wedge \mathbb{S} \rightarrow UL(G/H_+ \wedge \mathbb{S})$ is a $G^{\mathcal{U}}$ -stable equivalence.

We now see that this map is even an isomorphism of G -symmetric spectra. There is a composition of adjunctions:

$$\begin{array}{ccccc}
 & & \Sigma^\infty & & \\
 & \nearrow \Sigma^\infty & & \nwarrow L & \\
 G\mathbf{T}_* & \xrightarrow{\quad} & GSp_{\mathbf{T}}^\Sigma & \xleftarrow{\quad} & GSp^O \\
 & \nwarrow (-)_0 & & \nearrow U & \\
 & & (-)_0 & &
 \end{array}$$

In particular, L maps a symmetric suspension spectrum to the according orthogonal suspension spectrum. So does U the other way around, by definition. Furthermore, both suspension spectrum functors Σ^∞ are fully faithful and hence so is L when restricted to suspension spectra. It follows that the unit $\eta_{\Sigma^\infty A} : \Sigma^\infty A \rightarrow UL(\Sigma^\infty A)$ is an isomorphism for all based G -spaces A and in particular for $A = G/H_+$. \square

Proof of Theorem 5.2. Once we have shown that the adjunction is a Quillen pair, we can proceed as above or use 2-out-of-3 for Quillen equivalences, since we already know that our flat and projective model structures are Quillen equivalent and the same holds for G -orthogonal spectra ([Sto11, Proposition 2.3.31]).

The flat \mathcal{V} -level model structure on G -orthogonal spectra of Stolz is built in the same way as the one on G -symmetric spectra (since it was our role model), replacing the families $\mathcal{F}_{\mathcal{U}(\mathcal{V})}^{G, \Sigma_n}$ by their orthogonal analogs $\mathcal{F}_{\mathcal{V}}^{G, O(n)}$. The graph of a homomorphism $H \rightarrow \Sigma_n$ lies in $\mathcal{F}_{\mathcal{U}(\mathcal{V})}^{G, \Sigma_n}$ if and only if the graph of the composition $H \rightarrow \Sigma_n \rightarrow O(n)$ lies in $\mathcal{F}_{\mathcal{V}}^{G, O(n)}$, so we see that pulling

back $E\mathcal{F}_V^{G,O(n)}$ along $G \times \Sigma_n \rightarrow G \times O(n)$ gives a model for $E\mathcal{F}_{\mathcal{U}(V)}^{G,\Sigma_n}$ and it follows that U is a right Quillen functor for the level model structures. The fibrant objects of the stable model structure on G -orthogonal spectra are precisely the level fibrant $G^V\Omega$ -spectra, so we again see that U preserves fibrant objects of the stable model structures and can argue in the same way as above that the adjunction is a Quillen pair. This finishes the proof. \square

5.2 Quillen equivalence to Mandell's equivariant symmetric spectra

The adjunction to Mandell's equivariant symmetric spectra (cf. Section 2.4) also forms a Quillen equivalence. For a normal subgroup N of G we denote by $\mathcal{U}_{G/N}$ the G -set universe consisting of an infinite disjoint union of copies of G/N . It models the N -fixed G -equivariant stable homotopy category.

Theorem 5.3 (Quillen equivalence to $G\Sigma_{G/N}$ -spectra). *For every finite group G and every normal subgroup N the adjunction*

$$L_{G/N} : G\Sigma_{G/N} \rightleftarrows GSp_{\mathcal{S}}^{\Sigma} : U_{G/N}$$

is a Quillen equivalence for the stable model structure of [Man04, Theorem 4.1] on $G\Sigma_{G/N}$ and the $\mathcal{U}_{G/N}$ -projective stable model structure on $GSp_{\mathcal{S}}^{\Sigma}$.

Proof. The proof is similar to the one of Theorem 5.1. We first show that the adjunction is a Quillen pair. In the level model structure of [Man04, Definition 3.1] the weak equivalences and fibrations are defined levelwise, so it follows that the forgetful functor $U_{G/N}$ becomes a right Quillen functor for the $\mathcal{U}_{G/N}$ -projective level model structure on GSp^{Σ} . The stable model structure ([Man04, Theorem 4.1]) is obtained via a left Bousfield localization from the level model structure, so it has the same cofibrations and we can deduce that $L_{G/N}$ preserves cofibrations. The fibrant objects of the stable model structure are precisely those level fibrant X for which $X(n \times G/H) \rightarrow \Omega^{G/N} X((n+1) \times G/H)$ is a genuine G -equivalence, so we see that $U_{G/N}$ also preserves fibrant objects. Via the same argument as in Theorem 5.1 we deduce that the adjunction is a Quillen pair. In order to see that it is a Quillen equivalence we consider the chain of Quillen adjunctions

$$G\Sigma_{G/N} \xrightleftharpoons[U_{G/N}]{L_{G/N}} GSp_{\mathcal{S}, \mathcal{U}_{G/N}, \text{proj}}^{\Sigma} \xrightleftharpoons[\mathcal{S}]{|\cdot|} GSp_{\mathbf{T}, \mathcal{U}_{G/N}, \text{proj}}^{\Sigma} \xrightleftharpoons[U]{L} GSp_{\mathbb{R}[\mathcal{U}], \text{proj}}^O.$$

By [Man04, Theorem 10.2], the derived functors of the full composite are equivalences on the homotopy categories. We already know that the second and third adjunction are Quillen equivalences and hence also induce equivalences on the homotopy categories. It follows that so does the first, which finishes the proof. \square

Remark 5.4. The comparison between G -symmetric spectra and $G\Sigma_{G/\{e\}}$ -spectra is considered in more detail in [Ste14], emphasizing the point of view as equivariant functor categories. There, Steimle constructs another model structure on G -symmetric spectra which slightly differs from the projective one developed in this paper (with respect to the complete G -set universe) in that it only considers evaluations at finite G -sets (cf. the discussion after Lemma 2.28). This leads to less cofibrations and in [Ste14] it is shown that the functor $U_{G/\{e\}}$ takes these to cofibrations of $G\Sigma_{G/\{e\}}$ -spectra in the sense of [Man04]. In addition, a flat-type stable model structure on $G\Sigma_{G/\{e\}}$ -spectra is developed and proven to be Quillen equivalent to the one in [Man04].

6 Derived functors

In this section we explain how the model structures can be used to derive various functors between categories of equivariant symmetric spectra.

6.1 Change of universe

The \mathcal{U} -stable homotopy categories $\mathrm{Ho}^{\mathcal{U}}(GSp^{\Sigma})$ for varying G -set universes \mathcal{U} are related by so-called change of universe functors. For this we let $\mathcal{U}, \mathcal{U}'$ be G -set universes such that \mathcal{U} allows an embedding into \mathcal{U}' . The identity adjunction can be derived with respect to the $G^{\mathcal{U}}$ - and $G^{\mathcal{U}'}$ -stable equivalences in both directions, one of which is classical and one seems to not have appeared in the literature.

We start with the classical one. It follows immediately from the definitions that every $G^{\mathcal{U}'}$ -level equivalence is also a $G^{\mathcal{U}}$ -level equivalence and the same is true for projective level fibrations. Moreover, every $G^{\mathcal{U}'}$ - Ω -spectrum is in particular a $G^{\mathcal{U}}$ - Ω -spectrum and so we see that the identity functor is a right Quillen functor from the $G^{\mathcal{U}'}$ -projective stable model structure to the $G^{\mathcal{U}}$ -projective stable model structure. Hence, we obtain a Quillen pair

$$id : GSp_{\mathcal{U},proj}^{\Sigma} \rightleftarrows GSp_{\mathcal{U}',proj}^{\Sigma} : id \quad (6.1)$$

and a derived adjunction between the homotopy categories. We note that the Quillen pair does not depend on a choice of embedding $\mathcal{U} \hookrightarrow \mathcal{U}'$.

If the \mathbb{R} -linearizations of \mathcal{U} and \mathcal{U}' are isomorphic, it follows by 2-out-of-3 for the Quillen equivalences from Theorem 5.1 that the above adjunction is a Quillen equivalence. In fact, less is necessary: It is a consequence of [Lew95, Theorem 1.2], that we obtain a Quillen equivalence already if every G -orbit G/H which embeds into $\mathbb{R}[\mathcal{U}']$ also embeds into $\mathbb{R}[\mathcal{U}]$.

Remark 6.1. The identity adjunction is usually not a Quillen pair for the flat stable model structures, since a $G^{\mathcal{U}}$ -stable equivalence between G -flat G -symmetric spectra is not necessarily a $G^{\mathcal{U}'}$ -stable equivalence.

In order to derive the identity adjunction in the other direction (i.e., thinking of the identity from G -symmetric spectra with $G^{\mathcal{U}'}$ -stable equivalences to G -symmetric spectra with $G^{\mathcal{U}}$ -stable equivalences as the left adjoint) we need the flat model structures, though it does not quite become a Quillen pair with respect to them. We instead make use of the theory of deformations (cf. [DHKS04, Chapter VII], or [Rie14]), which we quickly recall. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of homotopical categories, i.e., categories with weak equivalences satisfying 2-out-of-6. Then a left deformation (called a “left deformation retract” in [DHKS04]) of F is an endofunctor $l : \mathcal{C} \rightarrow \mathcal{C}$ with a natural weak equivalence $l \rightarrow id_{\mathcal{C}}$ such that F is homotopical on the essential image of l , i.e., it sends all weak equivalences of the form $l(c) \rightarrow l(c')$ to weak equivalences in \mathcal{D} . In particular, the composition $F \circ l$ is homotopical and hence it induces a functor between the homotopy categories obtained by inverting the weak equivalences. By [DHKS04, 39.3], this yields a left derived functor of F .

Similarly, a right deformation for F is an endofunctor on whose image F is homotopical and which receives a natural weak equivalence from the identity. It can be used to compute right derived functors between the homotopy categories. Finally, an adjunction is said to be deformable if the left adjoint allows a left deformation and the right adjoint allows a right deformation. In this case the derived functors again form an adjunction ([DHKS04, 39.7]).

We now show that in the situation described above the identity adjunction is deformable. By $GSp_{\mathcal{U}}^{\Sigma}$ (resp. $GSp_{\mathcal{U}'}^{\Sigma}$) we mean the homotopical category of G -symmetric spectra equipped with $G^{\mathcal{U}}$ -stable equivalences (resp. $G^{\mathcal{U}'}$ -stable equivalences). Furthermore, given two G -sets M and N we let $\mathcal{F}(M, N)$ be the family of subgroups H of G for which M embeds H -equivariantly into N .

Proposition 6.2. *The identity adjunction $GSp_{\mathcal{U}'}^{\Sigma} \rightleftarrows GSp_{\mathcal{U}}^{\Sigma}$ is deformable and the derived functors are naturally isomorphic to those of the Quillen adjunction*

$$EF(\mathcal{U}', \mathcal{U})_+ \wedge (-) : GSp_{\mathcal{U}',flat}^{\Sigma} \rightleftarrows GSp_{\mathcal{U},flat}^{\Sigma} : \mathrm{map}(EF(\mathcal{U}', \mathcal{U})_+, -).$$

That the latter adjunction is a Quillen pair for the flat model structures follows from the remark after Proposition 4.1: The G -CW complex $EF(\mathcal{U}', \mathcal{U})_+$ has isotropy in $\mathcal{F}(\mathcal{U}', \mathcal{U})$ and hence smashing with it turns every map of G -symmetric spectra which is an $H^{\mathcal{U}}$ -equivalence for all H in $\mathcal{F}(\mathcal{U}', \mathcal{U})$ to a $G^{\mathcal{U}}$ -equivalence. But for all H in $\mathcal{F}(\mathcal{U}', \mathcal{U})$ the universes \mathcal{U} and \mathcal{U}' are isomorphic (since we also assumed that \mathcal{U} embeds into \mathcal{U}'), so every $G^{\mathcal{U}'}$ -stable equivalence has this property. Hence, smashing with $EF(\mathcal{U}', \mathcal{U})_+$ maps $G^{\mathcal{U}'}$ -stable equivalences to $G^{\mathcal{U}}$ -stable equivalences and it follows that the adjunction is a Quillen pair.

We start the proof of Proposition 6.2 by showing that the “right adjoint” $id : GSp_{\mathcal{U}}^{\Sigma} \rightarrow GSp_{\mathcal{U}'}^{\Sigma}$, is right deformable. We claim that a fibrant replacement $id \rightarrow (Q^{\mathcal{U}}-)^{bfib}$ in the flat \mathcal{U} -stable model structure does the job: Let $f : X \rightarrow Y$ be a $G^{\mathcal{U}}$ -stable equivalence between $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectra. Then by Remark 2.45 we know that f is a $G^{\mathcal{U}}$ -level equivalence, but since the levels of X and Y are cofree with respect to the families $EF_{\mathcal{U}}^{G, \Sigma_n}$ it follows that in fact each f_n is a genuine $(G \times \Sigma_n)$ -equivalence. In particular, f is a $G^{\mathcal{U}'}$ -level equivalence and hence a $G^{\mathcal{U}'}$ -stable equivalence, and so id is homotopical on the essential image.

The remainder of Proposition 6.2 follows from the following lemma, since for the G -set M of item (i) the families $\mathcal{F}(M, \mathcal{U})$ and $\mathcal{F}(\mathcal{U}', \mathcal{U})$ are equal. We recall that $\mathcal{F}_M S^M$ denotes the free spectrum on S^M in level M introduced in Section 2.7.

Lemma 6.3. (i) *Let M be the disjoint union over all orbits G/H which allow an embedding into \mathcal{U}' but not into \mathcal{U} . Then $\mathcal{F}_M S^M \wedge X \rightarrow X$ is a left deformation for the identity functor from G -symmetric spectra with $G^{\mathcal{U}'}$ -stable equivalences to G -symmetric spectra with $G^{\mathcal{U}}$ -stable equivalences.*

(ii) *For every finite G -set N the G -symmetric spectrum $\mathcal{F}_N S^N$ is $G^{\mathcal{U}}$ -stably equivalent to $\Sigma^{\infty} EF(N, \mathcal{U})_+$.*

Proof. We first show item (ii) by observing that they corepresent naturally isomorphic functors in the homotopy category $\text{Ho}^{\mathcal{U}}(GSp^{\Sigma})$. We express the hom-sets via the $G^{\mathcal{U}}$ -flat stable model structure and let Z be a $G^{\mathcal{U}}$ -flat level fibrant $G^{\mathcal{U}}\Omega$ -spectrum. We further note that both $\mathcal{F}_N S^N$ and $\Sigma^{\infty} EF(N, \mathcal{U})_+$ are G -flat. Then the natural isomorphism is given by the chain

$$\begin{aligned} \text{Ho}^{\mathcal{U}}(GSp^{\Sigma})(\Sigma^{\infty} EF(N, \mathcal{U})_+, Z) &\cong [EF(N, \mathcal{U})_+, Z_0]^G \\ &\cong [EF(N, \mathcal{U})_+, \Omega^N Z(N)]^G \\ &\cong [S^N, \text{map}(EF(N, \mathcal{U})_+, Z(N))]^G \\ &\cong [S^N, Z(N)]^G \\ &\cong \text{Ho}^{\mathcal{U}}(GSp^{\Sigma})(\mathcal{F}_N S^N, Z) \end{aligned}$$

Here, the second isomorphism uses that the map $Z_0 \rightarrow \Omega^N Z(N)$ is an $\mathcal{F}(N, \mathcal{U})$ -equivalence, since Z is a $G^{\mathcal{U}}\Omega$ -spectrum. The second to last isomorphism makes use of the fact that Z is $G^{\mathcal{U}}$ -flat level fibrant and hence $Z(N)$ is G -equivalent to $\text{map}(EF(N, \mathcal{U})_+, Z(N))$.

(i): It follows from Example 2.46 and Proposition 7.1 that $\mathcal{F}_M S^M \wedge X \rightarrow X$ is a natural $G^{\mathcal{U}'}$ -stable equivalence, since M is contained in \mathcal{U} . Hence, it remains to show that the identity is homotopical on spectra of the form $\mathcal{F}_M S^M \wedge X$. Let $f : \mathcal{F}_M S^M \wedge X \rightarrow \mathcal{F}_M S^M \wedge Y$ be a $G^{\mathcal{U}'}$ -stable equivalence. Then, as we argued above, the map $EF(\mathcal{U}', \mathcal{U})_+ \wedge f$ is a $G^{\mathcal{U}}$ -stable equivalence. But by (ii) we know that there exists a zig-zag of $G^{\mathcal{U}}$ -stable equivalences $\mathcal{F}_M S^M \rightarrow W \leftarrow \Sigma^{\infty} EF(M, \mathcal{U})_+ = \Sigma^{\infty} EF(\mathcal{U}', \mathcal{U})_+$ (where W can be chosen to also be G -flat). Furthermore, the projection $EF(\mathcal{U}', \mathcal{U})_+ \wedge EF(\mathcal{U}', \mathcal{U})_+ \rightarrow EF(\mathcal{U}', \mathcal{U})_+$ is a G -homotopy equivalence, and so it follows (by repeatedly applying Proposition 7.1), that the maps $EF(\mathcal{U}', \mathcal{U})_+ \wedge \mathcal{F}_M S^M \wedge X \rightarrow \mathcal{F}_M S^M \wedge X$ and $EF(\mathcal{U}', \mathcal{U})_+ \wedge \mathcal{F}_M S^M \wedge Y \rightarrow \mathcal{F}_M S^M \wedge Y$ are $G^{\mathcal{U}}$ -stable equivalences. Hence we see that f is also a $G^{\mathcal{U}}$ -stable equivalence, which finishes the proof. \square

The change of universe functors obtained this way behave differently to what one might expect, as the example below shows:

Example 6.4. Let G be a non-trivial finite group, \mathcal{U}_G be a complete G -set universe and $\mathcal{U}_{G/e}$ a free G -set universe. Then $\mathcal{U}_{G/e}$ embeds into \mathcal{U}_G and we can consider the two kinds of change of universe functors. Since the \mathbb{R} -linearizations of \mathcal{U}_G and $\mathcal{U}_{G/e}$ are isomorphic, we see that the “classical” change of universe adjunction (Adjunction 6.1 above) is a Quillen equivalence. On the other hand, the only subgroup H of G for which \mathcal{U}_G is isomorphic to $\mathcal{U}_{G/e}$ is the trivial group, so the left derived functor of $id : GSp_{\mathcal{U}_G}^\Sigma \rightarrow GSp_{\mathcal{U}_{G/e}}^\Sigma$ is given by smashing with EG_+ . This is far from being an equivalence on the homotopy categories, for example the sphere spectrum is not in the essential image.

In particular, the composition of the two left derived functors of $id : GSp_{\mathcal{U}_{G/e}}^\Sigma \rightarrow GSp_{\mathcal{U}_G}^\Sigma$ and $id : GSp_{\mathcal{U}_G}^\Sigma \rightarrow GSp_{\mathcal{U}_{G/e}}^\Sigma$ is *not* a left derived functor for the composition $id : GSp_{\mathcal{U}_{G/e}}^\Sigma \rightarrow GSp_{\mathcal{U}_{G/e}}^\Sigma$.

6.2 Change of groups

Let $H \leq G$ be a subgroup inclusion. Then the restriction functor $\text{res}_H^G : GSp^\Sigma \rightarrow HSp^\Sigma$ preserves all the structure one could ask for, namely it maps G -flat cofibrations ($G^\mathcal{U}$ -flat stable fibrations) to H -flat cofibrations (resp. $H^\mathcal{U}$ -flat stable fibrations) and all $G^\mathcal{U}$ -stable equivalences to $H^\mathcal{U}$ -stable equivalences, and similarly for the projective model structure and their positive versions. Hence, it is both a right and a left Quillen functor and one obtains Quillen pairs

$$G \ltimes_H (-) : HSp^\Sigma \rightleftarrows GSp^\Sigma : \text{res}_H^G$$

and

$$\text{res}_H^G : GSp^\Sigma \rightleftarrows HSp^\Sigma : \text{map}_H(G, -)$$

with respect to either the flat or the projective \mathcal{U} -stable model structures, positive and non-positive. The Wirthmüller isomorphism (Proposition 3.11) implies that if G/H embeds into $\mathbb{R}[\mathcal{U}]$, the derived functors $\mathbb{L}(G \ltimes_H -)$ and $\mathbb{R}(\text{map}_H(G, -))$ are naturally isomorphic.

6.3 Categorical fixed points

If \mathcal{U} contains a trivial subuniverse, the (trivial action/ G -fixed points) adjunction

$$\text{triv} : Sp^\Sigma \rightleftarrows GSp^\Sigma : (-)^G$$

becomes a Quillen pair for the $G^\mathcal{U}$ -projective model structure on GSp^Σ and the projective model structure on Sp^Σ . Hence, in order to compute the derived fixed points, which are often called “categorical fixed points”, one has to replace by a $G^\mathcal{U}\Omega$ -spectrum, and in order to left derive “equipping a symmetric spectrum with trivial G -action” one replaces by a projective one first. For every $G^\mathcal{U}\Omega$ -spectrum X there is a natural isomorphism $\pi_*(X^G) \cong \pi_*^{G,\mathcal{U}} X$, so in general the homotopy groups of the derived fixed points are isomorphic to the *true* (derived) equivariant homotopy groups.

Remark 6.5. The flat model structures are not suitable for deriving this adjunction (unless \mathcal{U} is the trivial universe), even though the “trivial action” functor preserves flat cofibrations: A stable equivalence between flat symmetric spectra is not necessarily a $G^\mathcal{U}$ -stable equivalence when equipped with the trivial action.

Remark 6.6. If \mathcal{U} does not contain a trivial subuniverse, the above adjunction is not a Quillen pair. However, the proof of Proposition 6.2 shows that $X \rightarrow Q^\mathcal{U} X^{bfib}$ is a right deformation for $(-)^G$, and hence the right derived functor can be computed as $(Q^\mathcal{U} X^{bfib})^G$. Alternatively, by Theorem 5.1 every G -symmetric spectrum can be replaced by an orthogonal $G^{\mathbb{R}[\mathcal{U}]}\Omega$ -spectrum up to $G^\mathcal{U}$ -stable equivalence, and fixed points of orthogonal $G^{\mathbb{R}[\mathcal{U}]}\Omega$ -spectra are always non-equivariant Ω -spectra, since $\mathbb{R}[\mathcal{U}]$ always contains a trivial subuniverse.

In case a G -symmetric spectrum is $G^{\mathcal{U}}$ -semistable, one can express the derived fixed points more directly. For this we choose a finite G -set M such that \mathcal{U} is isomorphic to a countable disjoint union of copies of M (i.e., M has to contain every G -orbit of \mathcal{U} and no others). We recall that $S^{\overline{M}}$ denotes the sphere in the reduced linearization of M , i.e., $\mathbb{R}[M]$ modulo the trivial diagonal copy of \mathbb{R} . Then for a G -symmetric spectrum of spaces X we define $F_{\mathcal{U}}^G X$ to be the non-equivariant symmetric spectrum with n -th level

$$(F_{\mathcal{U}}^G X)_n = \text{map}_G(S^{n \cdot \overline{M}}, X(n \times M))$$

with Σ_n -action via conjugation (permuting the n copies of M resp. \overline{M} on both sides). The structure map sends a pair $(f : S^{n \cdot \overline{M}} \rightarrow X(n \times M), x \in S^1)$ to the composite

$$S^{(n+1) \cdot \overline{M}} \xrightarrow{f \wedge S^{\overline{M}}} X(n \times M) \wedge S^{\overline{M}} \xrightarrow{- \wedge x} X(n \times M) \wedge S^M \xrightarrow{\sigma_{n \times M}^M} X((n+1) \times M).$$

By adjunction, there is a natural isomorphism of *naive* homotopy groups

$$\pi_k F_{\mathcal{U}}^G = \text{colim}_{n \in \mathbb{N}} [S^{k \sqcup n}, \text{map}_G(S^{n \times \overline{M}}, X(n \cdot M))] \cong \text{colim}_{n \in \mathbb{N}} [S^{k \sqcup (n \times M)}, X(n \times M)]^G \cong \pi_k^{G, \mathcal{U}} X.$$

This implies:

Proposition 6.7. *A map of G -symmetric spectra of spaces is a $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism if and only if it induces a non-equivariant π_* -isomorphism on fixed points $F_{\mathcal{U}}^H$ for all subgroups H of G .*

Moreover, a G -symmetric spectrum of spaces X is $G^{\mathcal{U}}$ -semistable if and only if all its fixed points $F_{\mathcal{U}}^H X$ are non-equivariantly semistable.

The latter is a consequence of the fact that the shift along M on $\pi_*^{G, \mathcal{U}} X$ corresponds to the shift along 1 on $\pi_* F_{\mathcal{U}}^G X$ under the isomorphism above.

Remark 6.8. It is possible to show that $F_{\mathcal{U}}^G X$ only depends on M up to π_* -isomorphism, by comparing two possible choices M and M' to their disjoint union $M \sqcup M'$.

In the case where \mathcal{U} (and hence M) has non-empty G -fixed points, there is a natural map $X^G \rightarrow F_{\mathcal{U}}^G X$ given in level n by applying fixed points to the adjoint structure map $X_n \rightarrow \text{map}(S^{nM}, X(nM))$ associated to a decomposition $M = \{1\} \sqcup \overline{M}$. If X is a $G^{\mathcal{U}}\Omega$ -spectrum, this map is a level-equivalence, so since $F_{\mathcal{U}}^G$ preserves $\underline{\pi}_*^{\mathcal{U}}$ -isomorphisms we see:

Proposition 6.9. *The functor $F_{\mathcal{U}}^G$ computes the right derived functor of categorical fixed points on $G^{\mathcal{U}}$ -semistable spectra.*

If \mathcal{U} has empty G -fixed points this still remains true, which is easiest to show via G -orthogonal spectra: Since $\mathbb{R}[\mathcal{U}]$ always contains a trivial subuniverse there is a natural map $X^G \rightarrow F_{\mathcal{U}}^G X$ for every G -orthogonal spectrum X , which is a level equivalence if X is a $G^{\mathbb{R}[\mathcal{U}]}\Omega$ -spectrum. Since we have shown G -orthogonal spectra to be Quillen equivalent to G -symmetric spectra (Theorem 5.1), every $G^{\mathcal{U}}$ -semistable G -symmetric spectrum can be replaced by an orthogonal $G^{\mathbb{R}[\mathcal{U}]}\Omega$ -spectrum up to $\underline{\pi}_*^{\mathcal{U}}$ -isomorphism, which proves the claim.

Remark 6.10. A simplicial version of $F_{\mathcal{U}}^G$ is obtained by either restricting to $G^{\mathcal{U}}$ -level fibrant G -symmetric spectra or by putting in a suitable fibrant replacement functor (such as $\mathcal{S}[\cdot]$) into the target of the mapping space. In either case, the analogs of Propositions 6.7 and 6.9 hold.

In general, $F_{\mathcal{U}}^G X$ does not compute the derived fixed points if X is not $G^{\mathcal{U}}$ -semistable, as Example 6.14 shows.

6.4 Geometric fixed points

Now we deal with geometric fixed points. Let EP denote a universal spaces for the family of proper subgroups of G and \tilde{EP} its unreduced suspension, i.e., the cone of the map $EP_+ \rightarrow S^0$ collapsing EP to a point. Then the G -fixed points of \tilde{EP} are isomorphic to S^0 and the fixed points for all proper subgroups are contractible.

Definition 6.11 (Geometric fixed points). The geometric fixed points $\Phi_{\mathcal{U}}^G X$ of a G -symmetric spectrum X are defined as $(\tilde{EP} \wedge X)^{\mathbb{R}G}$, the derived fixed points of $\tilde{EP} \wedge X$.

Again this can be made more explicit by using the fixed point construction $F_{\mathcal{U}}^G$ of the previous section. That $F_{\mathcal{U}}^G(\tilde{EP} \wedge -)$ computes the geometric fixed points on $G^{\mathcal{U}}$ -semistable G -symmetric spectra is a consequence of the following proposition, which also shows that geometric fixed points detect $G^{\mathcal{U}}$ -stable equivalences:

Proposition 6.12. *The following hold:*

- (i) *A map of G -symmetric spectra $f : X \rightarrow Y$ is a $G^{\mathcal{U}}$ -stable equivalence if and only if $\Phi_{\mathcal{U}}^H f$ is a stable equivalence for all subgroups H of G .*
- (ii) *A map of G -symmetric spectra of spaces $f : X \rightarrow Y$ is a $\pi_*^{\mathcal{U}}$ -isomorphism if and only if $F_{\mathcal{U}}^H(\tilde{EP} \wedge f)$ is a π_* -isomorphism for all subgroups H of G .*
- (iii) *A G -symmetric spectrum of spaces X is $G^{\mathcal{U}}$ -semistable if and only if $F_{\mathcal{U}}^H(\tilde{EP} \wedge f)$ is non-equivariantly semistable for all subgroups H of G .*

Again, simplicial versions of items (ii) and (iii) can be obtained as in Remark 6.10. We note that in order to compute $\Phi_{\mathcal{U}}^H X$ for a proper subgroup H of G one has to take the categorical fixed points of the H -symmetric spectrum $\tilde{EP} \wedge X$, where this time \tilde{EP} is formed with respect to proper subgroups of H (which is *not* the restriction of the \tilde{EP} for G).

Proof. The proofs of all the three parts proceed in the same way, we only explain the second one. Since smashing with a G -CW complex preserves $\pi_*^{\mathcal{U}}$ -isomorphisms, cf. Corollary 3.14, it follows that a $\pi_*^{\mathcal{U}}$ -isomorphism induces a non-equivariant π_* -isomorphism on all geometric fixed points. For the other direction we assume that $f : X \rightarrow Y$ induces a π_* -isomorphism on $F_{\mathcal{U}}^H$ for all subgroups H of G . We show by induction on the order of G that f is a $\pi_*^{\mathcal{U}}$ -isomorphism. Since \tilde{EP}_+ is a point for G the trivial group, the induction start is clear. So we can assume by the induction hypothesis that f induces an isomorphism on $\pi_*^{H, \mathcal{U}}$ for all proper subgroups H of G . Since EP_+ has isotropy in proper subgroups, Corollary 3.14 implies that $EP_+ \wedge f$ is a $\pi_*^{\mathcal{U}}$ -isomorphism. by assumption we also know that $\tilde{EP} \wedge f$ is a π_* -isomorphism, because \tilde{EP} is H -equivariantly contractible for all proper subgroups H of G and we know by assumption (using the natural isomorphism $\pi_* F_{\mathcal{U}}^H(\tilde{EP} \wedge (-)) \cong \pi_*^{G, \mathcal{U}}(\tilde{EP} \wedge (-))$) that $\tilde{EP} \wedge f$ induces an isomorphism on $\pi_*^{G, \mathcal{U}}$. So the induction step follows from the five-lemma applied to the map of long exact sequences of homotopy groups obtained by smashing f with the cofiber sequence $EP_+ \rightarrow S^0 \rightarrow \tilde{EP}$. \square

The symmetric spectrum $F_{\mathcal{U}}^H(\tilde{EP} \wedge X)$ can be described differently: Since the fixed points of \tilde{EP} along proper subgroups are all contractible, it follows from the equivariant Whitehead theorem (Proposition 1.19) that the “fixed point map”

$$\mathrm{map}_G(S^{n, \overline{M}}, \tilde{EP} \wedge X(n \times M)) \rightarrow \mathrm{map}(S^{n, \overline{M}^G}, \tilde{EP}^G \wedge X(n \times M)^G) \cong \mathrm{map}(S^{n, \overline{M}^G}, X(n \times M)^G)$$

is a weak equivalence. The latter has a particularly nice form if M can be chosen to be a transitive G -set, i.e., an orbit G/H , as in that case $S^{\overline{M}^G}$ is simply S^0 .

Corollary 6.13. *If \mathcal{U} is isomorphic to a countable disjoint union of copies of G/H for some subgroup H , then $F_{\mathcal{U}}^H(\tilde{E}P \wedge X)$ is naturally level-equivalent to the symmetric spectrum with n -th level $X(n \times G/H)^G$ and structure map the fixed points of the generalized structure map $\sigma_{n \times G/H}^{G/H}$ of X .*

We close with an example that $F_{\mathcal{U}}^G(\tilde{E}P \wedge X)$ does in general not compute the (derived) geometric fixed points if X is not $G^{\mathcal{U}}$ -semistable.

Example 6.14. Let G be a non-trivial finite group, $\mathcal{U}_{G/e}$ the free G -set universe $\mathbb{N} \times G$ and $\mathcal{F}_G S^G$ the free spectrum on S^G in level G . By Example 2.46 there is a $G^{\mathcal{U}_{G/e}}$ -stable equivalence $\lambda_{\emptyset, G} : \mathcal{F}_G S^G \rightarrow \mathbb{S}$. Since the sphere spectrum is $G^{\mathcal{U}_{G/e}}$ -semistable, the symmetric spectrum $F_{\mathcal{U}}^H(\tilde{E}P \wedge \mathbb{S})$ computes the correct geometric fixed points and Corollary 6.13 shows that they are level-equivalent to the non-equivariant sphere spectrum. On the other hand, again using Corollary 6.13 and Example 3.35 one sees that the n -th level of $F_{\mathcal{U}}^G(\tilde{E}P \wedge \mathcal{F}_G S^G)$ is weakly equivalent to $\text{Inj}(G, n \times G)_+^G \wedge (S^{n \times G})^G$. The fixed points $\text{Inj}(G, n \times G)^G$ equal the set of G -linear injections from G into $n \times G$, which can be identified with the set $n \times G$, with Σ_n permuting the n -copies of G . This can be Σ_n -equivariantly rewritten as $G_+ \wedge (\text{Inj}(1, n)_+ \wedge S^n)$ and we see that $F_{\mathcal{U}}^G(\tilde{E}P \wedge \mathcal{F}_G S^G)$ is level equivalent to $G_+ \wedge \mathcal{F}_1 S^1$, which in turn is stably equivalent to $G_+ \wedge \mathbb{S}$, a wedge of $|G|$ many copies of the sphere spectrum. If G is not the trivial group, this is not stably equivalent to the sphere spectrum itself, hence it does not compute the correct derived geometric fixed points.

6.5 Orbits

Let \mathbb{N} denote the trivial G -set universe. Then the (G -orbits/trivial action) adjunction

$$(-)/G : GSp^{\Sigma} \rightleftarrows Sp^{\Sigma} : \text{triv}$$

is a Quillen pair for the $G^{\mathbb{N}}$ -projective model structure on the left and the projective model structure on the right, and likewise for the flat model structures, as one checks directly from the definitions. The main input is that a non-equivariant Ω -spectrum X is already a $G^{\mathbb{N}}$ - Ω -spectrum, and in addition $G^{\mathbb{N}}$ -flat level fibrant if and only if it is non-equivariantly flat level fibrant.

We can use the results of Section 6.1 to show that this adjunction can also be derived with respect to $G^{\mathcal{U}}$ -stable equivalences for non-trivial universes \mathcal{U} , as long as \mathcal{U} still contains a trivial subuniverse. We let $\mathcal{F}(\mathcal{U}, \mathbb{N})$ denote the family of subgroups of G which act trivially on \mathcal{U} . Then:

Proposition 6.15. *For every G -set universe \mathcal{U} containing a trivial subuniverse, the adjunction*

$$(-)/G : GSp_{\mathcal{U}}^{\Sigma} \rightleftarrows Sp^{\Sigma} : \text{triv}$$

is deformable and the derived adjunction is naturally isomorphic to the derived adjunction of the Quillen pair

$$EF(\mathcal{U}, \mathbb{N})_+ \wedge_G (-) : GSp_{\mathcal{U}, \text{flat}}^{\Sigma} \rightleftarrows Sp_{\text{flat}}^{\Sigma} : \text{map}(EF(\mathcal{U}, \mathbb{N})_+, -)$$

Proof. Lemma 6.3 (together with the fact that $(-)/G$ takes $G^{\mathbb{N}}$ -stable equivalences between G -flat G -symmetric spectra to non-equivariant stable equivalences) implies that $\mathcal{F}_M S^M \wedge X^b \rightarrow X^b \rightarrow X$ is a left deformation for $(-)/G$, where M is the disjoint union of all non-trivial G -orbits embedding into \mathcal{U} . Furthermore, by the proof of Proposition 6.2, any fibrant replacement in the flat stable model structure is a right deformation for the trivial action functor. Again by Lemma 6.3, $\mathcal{F}_M S^M \wedge X^b$ is $G^{\mathbb{N}}$ -stably equivalent to $EF(\mathcal{U}, \mathbb{N})_+ \wedge X^b$, hence (as both of them are flat) their orbits are non-equivariantly stably equivalent. \square

In particular, if \mathcal{U}_G is a complete G -set universe, the derived orbits $\mathbb{L}(-)/G : \text{Ho}^{\mathcal{U}_G}(GSp^{\Sigma}) \rightarrow \text{Ho}(Sp^{\Sigma})$ are naturally isomorphic to the homotopy orbits.

7 Monoidal properties

In this section we expand the relationship between the smash product and the stable model structures, constructing model structures on categories of modules, algebras (both via [SS00]) and commutative algebras (using [Whi14]). We also explain homotopical properties of the multiplicative norm.

7.1 The monoid axiom and model structures on module and algebra categories

We already know that our stable model structures are monoidal (Proposition 4.13), but in order to obtain model structures on modules and algebras over an arbitrary G -symmetric ring spectrum we need to show one more property called the monoid axiom (cf. [SS00, Def 3.3]). The main ingredient is the following:

Proposition 7.1 (Flatness). *(i) Smashing with a G -flat G -symmetric spectrum preserves $\pi_*^{\mathcal{U}}$ -isomorphisms and $G^{\mathcal{U}}$ -stable equivalences.*

(ii) Smashing with any G -symmetric spectrum preserves $\pi_^{\mathcal{U}}$ -isomorphisms and $G^{\mathcal{U}}$ -stable equivalences between G -flat G -symmetric spectra.*

Before we come to showing this we need one lemma. We recall that $\mathcal{G}_n(-)$ denotes the semi-free G -symmetric spectrum functor in level n (cf. Section 2.7).

Lemma 7.2. *Let $n \in \mathbb{N}$, A a cofibrant $(G \times \Sigma_n)$ -space and Y a G -symmetric spectrum with $\pi_*^{\mathcal{U}} Y = 0$. Then $\pi_*^{\mathcal{U}}(\mathcal{G}_n(A) \wedge Y) = 0$.*

Proof. It suffices to prove the lemma for G -symmetric spectra of spaces. From Section 2.6 we know that the $(n+m)$ -th level of the G -symmetric spectrum $\mathcal{G}_n(A) \wedge Y$ is given by $\Sigma_{n+m} \ltimes_{\Sigma_n \times \Sigma_m} (A \wedge Y_m)$. We let $\varphi : G \rightarrow \Sigma_{n+m}$ be a homomorphism such that the associated G -set $n + m_{\varphi}$ (which we denote by M from now on) embeds into \mathcal{U} , and $f : S^{k \sqcup M} \rightarrow (\mathcal{G}_n(A) \wedge Y)(M)$ a G -map. We have to show that f is stably null-homotopic. Applying the double coset formula (cf. Section 1.1), we see that $(\mathcal{G}_n(A) \wedge Y)(M)$ splits off as

$$\bigvee_{[N \subseteq M = n+m, |N|=n]} G \ltimes_{\text{Stab}(N)} (A \wedge Y(M - N)),$$

where the wedge is taken over a system of G -orbit representatives N of subsets of M of cardinality n and $\text{Stab}(N) \subseteq G$ is the stabilizer of such a subset. The stabilizer fixes both N and the complement $M - N$, so it acts on $Y(M - N)$. The action on A comes from pulling back the $(G \times \Sigma_n)$ -action along the graph of the homomorphism $\text{Stab}(f) \rightarrow \Sigma_N \cong \Sigma_n$ (where the latter is induced by the canonical order-preserving bijection $\underline{n} \cong N \subseteq \underline{n+m}$). We now fix such an n -elemental subset N and consider the G -map $f' : S^{k \sqcup M} \rightarrow (\mathcal{G}_n(A) \wedge Y)(M) \rightarrow G \ltimes_{\text{Stab}(N)} (A \wedge Y(M - N))$, i.e., f followed by the projection to this summand. It represents an element in $\pi_k^{G, \mathcal{U}}(\Omega^M(G \ltimes_{\text{Stab}(N)} (A \wedge Y)))$, which is trivial by Lemma 3.13 and Corollary 3.14. Hence, there exists a finite G -set M' such that $\sigma_{M-N}^{M'} \circ (f' \wedge S^{M'})$ is G -nullhomotopic. Taking M' large enough so that this holds for all subsets N at once, we obtain that $\sigma_M^{M'} \circ (f \wedge S^{M'})$ has the property that all postcompositions with projections to the summands are G -nullhomotopic. Using Corollary 3.10 we see that this implies that a suspension of f itself is G -nullhomotopic and so we are done. \square

Proof of Proposition 7.1. The simplicial case follows from the topological one, so we only have to show the latter. We first prove the statement about homotopy groups. By the long exact

sequence of the mapping cone (Proposition 3.8) it suffices to show that $X \wedge Y$ has trivial homotopy groups whenever X is G -flat and Y has trivial homotopy groups. We now prove by induction over n that $F^n X \wedge Y$ has trivial homotopy groups (where $F^n X$ denotes the n -skeleton of X , cf. Section 2.8). Since X is G -flat the maps $F^n X \wedge Y \rightarrow F^{n+1} X \wedge Y$ are h -cofibrations and so we can pass to the colimit to obtain the desired result. The induction start follows from $F^{-1} X = *$. Hence, for the induction step we assume shown that $F^{n-1} X \wedge Y$ has trivial homotopy groups and want to show that so does $F^n X \wedge Y$. We consider the pushout square (cf. Section 2.8):

$$\begin{array}{ccc} \mathcal{G}_n(L_n X) \wedge Y & \xrightarrow{\varepsilon_{F^{n-1} X} \wedge Y} & F^{n-1} X \wedge Y \\ \mathcal{G}_n(\nu_n X) \wedge Y \downarrow & & \downarrow (j_n X) \wedge Y \\ \mathcal{G}_n(X_n) \wedge Y & \xrightarrow{\varepsilon_{F^n X} \wedge Y} & F^n X \wedge Y \end{array}$$

By the induction hypothesis we know that the upper right term has trivial homotopy groups. Moreover, Lemma 7.2 implies that both terms on the left have trivial homotopy groups. The vertical maps in this square are h -cofibrations and so a comparison of the associated long exact sequences shows that the pushout $F^n X \wedge Y$ also has trivial homotopy groups.

For the statement about $G^{\mathcal{U}}$ -stable equivalences it again suffices to show that smashing with a G -flat symmetric spectrum preserves $G^{\mathcal{U}}$ -stably contractible G -symmetric spectra, since $G^{\mathcal{U}}$ -stable equivalences are detected by the mapping cone (Proposition 4.2). So we let Y be $G^{\mathcal{U}}$ -stably contractible, $i_Y : Y^b \rightarrow Y$ a G -flat replacement up to $G^{\mathcal{U}}$ -level equivalence and X an arbitrary G -flat G -symmetric spectrum. Then monoidality of the model structure (Proposition 4.13) implies that $X \wedge Y^b$ is $G^{\mathcal{U}}$ -stably contractible. The homotopy fiber of i_Y is even $G^{\mathcal{U}}$ -level contractible, hence it has trivial homotopy groups and so by the discussion above its smash product with X also has trivial homotopy groups. In particular, it is $G^{\mathcal{U}}$ -stably contractible. Thus, $X \wedge Y$ is also $G^{\mathcal{U}}$ -stably contractible and we are done.

Statement (ii) follows by 2-out-of-3 applied to a flat replacement. \square

Now we come to the monoid axiom. For a G -symmetric spectrum Y we denote by $\{J_{st}^{\mathcal{U}} \wedge Y\}^{cell}$ the class of maps obtained via (transfinite) compositions and pushouts from maps of the form $j \wedge Y$, where j is a G -flat cofibration and $G^{\mathcal{U}}$ -stable equivalence.

Proposition 7.3 (Monoid axiom). *Every map in $\{J_{st}^{\mathcal{U}} \wedge Y\}^{cell}$ is a $G^{\mathcal{U}}$ -stable equivalence.*

Proof. By Proposition 7.1 we know that each map $j \wedge Y$ is a $G^{\mathcal{U}}$ -stable equivalence and an h -cofibration. Since the class of $G^{\mathcal{U}}$ -stable equivalences which are also h -cofibrations is closed under (transfinite) composition and pushouts, this gives the monoid axiom. \square

This implies the monoid axiom [SS00, Def. 3.3] in all the positive and non-positive, projective and flat model structures, since every cofibration there is in particular a G -flat cofibration. Hence, by [SS00, Thm. 4.1] we obtain various model structures on the categories of modules and algebras, where the weak equivalences (and fibrations) are those maps which forget to a weak equivalence (resp. fibration) in the respective model structure on G -symmetric spectra.

Corollary 7.4. (i) *For every G -symmetric ring spectrum R the positive and non-positive, $G^{\mathcal{U}}$ -flat and $G^{\mathcal{U}}$ -projective stable model structures lift to the category of R -modules. If R is commutative, they are again monoidal.*

(ii) *For every commutative G -symmetric ring spectrum R the positive and non-positive, $G^{\mathcal{U}}$ -flat and $G^{\mathcal{U}}$ -projective stable model structures lift to the category of R -algebras. Furthermore, every cofibration of R -algebras with cofibrant source is also a cofibration of R -modules.*

Remark 7.5. Strictly speaking, Theorem 4.1 of [SS00] does not apply verbatim to the topological case, because not every G -symmetric spectrum of spaces is small. However, the domains and targets of the generating (acyclic) cofibrations are small with respect to transfinite compositions of flat cofibrations, and so the small object argument can nevertheless be applied (cf. [SS00, Remark 2.4]).

Proposition 7.1 also allows us to prove the following:

Proposition 7.6. *Let X and Y be two $G^{\mathcal{U}}$ -semistable G -symmetric spectra such that X is G -flat. Then $X \wedge Y$ is again $G^{\mathcal{U}}$ -semistable.*

Proof. We can reduce to the topological case, which is easiest to prove using G -orthogonal spectra. By Theorem 5.1, every $G^{\mathcal{U}}$ -semistable G -symmetric spectrum allows a zig-zag of $\pi_*^{\mathcal{U}}$ -isomorphisms to a $G^{\mathbb{R}[\mathcal{U}]}$ -projective G -orthogonal spectrum. These $\pi_*^{\mathcal{U}}$ -isomorphisms are preserved by smashing with X (Proposition 7.1) and so we can without loss of generality assume that Y is the restriction of a $G^{\mathbb{R}[\mathcal{U}]}$ -projective G -orthogonal spectrum. We can further reduce to the case where Y is a free G -orthogonal spectrum $\mathcal{F}_V^O A$ over a based G -CW complex A and a G -subrepresentation V of $\mathbb{R}[\mathcal{U}]$, since any $G^{\mathbb{R}[\mathcal{U}]}$ -projective G -orthogonal spectrum can be built by iteratively attaching cells of this form. Now $X \wedge \mathcal{F}_V^O A$ is $G^{\mathcal{U}}$ -semistable if and only if $S^V \wedge X \wedge \mathcal{F}_V A \cong A \wedge X \wedge \mathcal{F}_V S^V$ is $G^{\mathcal{U}}$ -semistable, since smashing with S^V preserves and detects $G^{\mathcal{U}}$ -semistability (the latter follows from the natural $\pi_*^{\mathcal{U}}$ -isomorphism $id \rightarrow \Omega^V(S^V \wedge -)$, Proposition 3.6). But there is a $\pi_*^{\mathcal{U}}$ -isomorphism $\mathcal{F}_V S^V \rightarrow \mathbb{S}$ ([MM02, Lemma 4.5]), which again is preserved by smashing with X and hence we are left to show that $A \wedge X$ is $G^{\mathcal{U}}$ -semistable, which is Corollary 3.41. \square

7.2 Homotopical properties of the norm

In this section we deal with the homotopical properties of the norm functor, a multiplicative version of induction whose analog on G -orthogonal spectra plays a major role in the solution of the Kervaire invariant one problem in [HHR14], where its properties are studied in detail (Sections A.4 and B.5). An analysis of the norm is also needed for the construction of model structures on strictly commutative G -symmetric ring spectra, as will become apparent in the next section.

The norm can be constructed as follows: We fix a subgroup inclusion $H \leq G$. For every H -symmetric spectrum X and every $n \in \mathbb{N}$, the n -fold smash power $X^{\wedge n}$ has a natural action of the wreath product $\Sigma_n \wr H$, i.e., the semidirect product of Σ_n and H^n associated to the Σ_n -action permuting the factors. Taking n to be the index of H in G , any choice of a system of coset representatives $g_1, \dots, g_n \in G$ of G/H gives rise to an embedding $\varphi : G \rightarrow \Sigma_n \wr H; g \mapsto (\sigma(g), h_1(g), \dots, h_n(g))$ characterized by the formula $g \cdot g_i = g_{\sigma(g)(i)} h_i(g)$. This embedding is independent of the chosen g_i up to conjugation. The norm is then defined as the G -symmetric spectrum obtained by restricting $X^{\wedge n}$ along this embedding.

The norm functor is symmetric monoidal, hence it maps commutative H -symmetric ring spectra to commutative G -symmetric ring spectra, yielding a left adjoint of the restriction functor.

We now summarize the homotopical properties of these constructions. They are very clean over the projective model structures. Given an H -set universe \mathcal{U} , the n -fold disjoint union $n \times \mathcal{U}$ becomes a $(\Sigma_n \wr H)$ -set universe and we have:

Theorem 7.7. *For every H -set universe \mathcal{U} and every natural number n the functor $(-)^{\wedge n} : HSp^{\Sigma} \rightarrow (\Sigma_n \wr H)Sp^{\Sigma}$ maps $H^{\mathcal{U}}$ -stable equivalences between $H^{\mathcal{U}}$ -projective H -symmetric spectra to $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalences of $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective spectra. In particular, the norm $N_H^G : HSp^{\Sigma} \rightarrow GSp^{\Sigma}$ maps $G^{\mathcal{U}}$ -stable equivalences between $H^{\mathcal{U}}$ -projective H -symmetric spectra to $G^{G \ltimes_H \mathcal{U}}$ -stable equivalences.*

In particular, both the n -fold smash power and the norm can always be left derived, as any $H^{\mathcal{U}}$ -projective replacement functor is a left deformation for them.

The behavior with respect to the flat model structure is more delicate. It is *not* true that the functor $(-)^{\wedge n} : HSp^{\Sigma} \rightarrow (\Sigma_n \wr H)Sp^{\Sigma}$ takes $H^{\mathcal{U}}$ -stable equivalences between H -flat H -symmetric spectra to $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalences, in fact not even to $(\Sigma_n \times H)^{n \times \mathcal{U}}$ -stable equivalences. Nevertheless, under some conditions on the H -set universe \mathcal{U} , the norm functor is still homotopical on H -flat H -symmetric spectra. The precise conditions on \mathcal{U} needed are quite complicated and for simplicity we restrict to the case where \mathcal{U} is the full N -fixed H -set universe associated to a subgroup N of H which is normal in G . We denote this H -set universe, an infinite disjoint union of all orbits H/K with $N \leq K$, by $\mathcal{U}_H(N)$. In particular, the case where N is trivial gives the complete H -set universe. We then have:

Theorem 7.8. *Let $N \leq H \leq G$ be subgroups with N normal in G . Then the norm functor $N_H^G : HSp^{\Sigma} \rightarrow GSp^{\Sigma}$ takes $H^{\mathcal{U}_H(N)}$ -stable equivalences between H -flat H -symmetric spectra to $G^{\mathcal{U}_G(N)}$ -stable equivalences of G -flat G -symmetric spectra.*

In particular, for universes of this form the effect of the derived norm can be computed on H -flat H -symmetric spectra. The point in this additional work will become clear in the next section, where it is used to show that the derived norm of a commutative H -symmetric ring spectrum is equivalent as a G -symmetric spectrum to the derived norm of the underlying H -symmetric spectrum (cf. [HHR14, Section B.8]).

In the proofs of Theorems 7.7 and 7.8 we will make use of the distributive law of [HHR14, Section A.3.3.], which says that both the n -fold smash power $(\bigvee_I X_i)^{\wedge n}$ and the norm $N_H^G(\bigvee_I X_i)$ of a wedge $\bigvee_I X_i$ can be rewritten as a wedge of induced smash powers respectively norms. It is a consequence of the fact that the wedge distributes over the smash product. We quickly recall it in the versions we need and start with the case of smash powers. For this we choose a linear ordering on the index set I . Given a monotone function $f : \underline{n} \rightarrow I$, we let a_1, \dots, a_k be the different values of f and $n_i \in \mathbb{N}_{>0}$ the order of their respective preimages. Then the distributive law asserts that there is a natural $\Sigma_n \wr H$ -equivariant decomposition

$$\left(\bigvee_{i \in I} X_i \right)^{\wedge n} \cong \bigvee_{f : \underline{n} \rightarrow I, \text{ monotone}} \Sigma_n \wr H \ltimes_{\Sigma_{n_1} \wr H \times \dots \times \Sigma_{n_k} \wr H} (X_{a_1}^{\wedge n_1} \wedge \dots \wedge X_{a_k}^{\wedge n_k}).$$

In the case of the norm, we need to replace monotone functions $\underline{n} \rightarrow I$ by G -orbits of all functions $f : G/H \rightarrow I$. Given such an f , we let K be the stabilizer and g_1, \dots, g_l a system of representatives for double cosets $K \backslash G/H$. Then there is a G -isomorphism

$$N_H^G\left(\bigvee_{i \in I} X_i\right) \cong \bigvee_{[f : G/H \rightarrow I]} G \ltimes_K \left(\bigwedge_{i=1}^k N_{K \cap g_i H g_i^{-1}} c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H X_{f(g_i)} \right).$$

We now move towards the proofs of Theorems 7.7 and 7.8 and start with the following:

Proposition 7.9. *Let G and K be finite groups, \mathcal{U}_G a G -set universe, \mathcal{U}_K a K -set universe, $i : A \rightarrow B$ a $G^{\mathcal{U}_G}$ -projective cofibration and $j : X \rightarrow Y$ a $K^{\mathcal{U}_K}$ -projective cofibration. Then the pushout product $i \square j$ is a $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -projective cofibration. If either i or j is positive, so is $i \square j$. If i is a $G^{\mathcal{U}_G}$ -projective cofibration and $G^{\mathcal{U}_G}$ -stable equivalence and j is a $K^{\mathcal{U}_K}$ -projective cofibration, then $i \square j$ is a $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -stable equivalence.*

Proof. The statement about cofibrations can be checked on generating cofibrations, where it follows easily using the natural $(G \times K)$ -isomorphism $\mathcal{F}_M(-) \wedge \mathcal{F}_N(-) \cong \mathcal{F}_{M \sqcup N}(- \wedge -)$.

Hence, if i is a $G^{\mathcal{U}}$ -stable equivalence and $G^{\mathcal{U}_G}$ -projective cofibration and j is a $K^{\mathcal{U}_K}$ -projective cofibration, we already know that $i \square j$ is a $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -projective cofibration and

it suffices to show that the $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -projective quotient $B/A \wedge Y/X$ is $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -stably contractible. We thus fix a $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ - Ω -spectrum Z and have to show that $[B/A \wedge Y/X, Z]^{G \times K} \cong [B/A, \text{hom}(Y/X, Z)^K]^{G \times K}$ is trivial. The K -symmetric spectrum Y/X is \mathcal{U}_K -projective, hence it is $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ -projective when given the trivial G -action and so Corollary 2.43 implies that $\text{hom}(Y/X, Z)$ is again a level-fibrant $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ - Ω -spectrum. Since B/A is $G^{\mathcal{U}}$ -stably contractible and $G^{\mathcal{U}_G}$ -projective, it hence suffices to show that the K -fixed points of every $(G \times K)^{\mathcal{U}_G \sqcup \mathcal{U}_K}$ - Ω -spectrum are a $G^{\mathcal{U}_G}$ - Ω -spectrum, which follows from the definition. \square

Remark 7.10. This proposition does not hold over the flat model structures.

Theorem 7.7 is then a consequence of the following proposition, via Ken Brown's Lemma:

Proposition 7.11. *Let $f : X \rightarrow Y$ be an $H^{\mathcal{U}}$ -projective cofibration of $H^{\mathcal{U}}$ -projective H -symmetric spectra and $n \in \mathbb{N}$. Then $f^{\wedge n} : X^{\wedge n} \rightarrow Y^{\wedge n}$ is a projective $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -cofibration. If f is in addition an $H^{\mathcal{U}}$ -stable equivalence, then $f^{\wedge n}$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalence.*

Proof. It suffices to prove the topological case. We start by showing the claim for the generating (stable acyclic) cofibrations. The generating cofibrations are of the form $H \ltimes_K \mathcal{F}_M(i : A \rightarrow B)$ for i a genuine K -cofibration of based K -CW complexes. Then $(H \ltimes_K \mathcal{F}_M(i))^{\wedge n} \cong (\Sigma_n \wr H) \ltimes_{\Sigma_n \wr K} (\mathcal{F}_{M^n}(i^{\wedge n}))$, which is an $H^{n \times \mathcal{U}}$ -projective cofibration, since $i^{\wedge n}$ is a genuine $(\Sigma_n \wr K)$ -cofibration (and, for later reference, so is the n -fold pushout product $i^{\square n}$). Furthermore, if i is also a genuine K -equivalence, then $i^{\wedge n}$ is a genuine $(\Sigma_n \wr K)$ -equivalence. Hence, it remains to consider the generating stable acyclic cofibrations added from the level to the stable model structure. They are of the form $H \ltimes_K (\lambda_{M,N}^{(K)} : F_{M \sqcup N} S^N \rightarrow F_M S^0)$ followed by an H -homotopy equivalence. Hence, it suffices to show that $(H \ltimes_K \lambda_{M,N}^{(K)})^n$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalence. But this follows from the identification $(H \ltimes_K \lambda_{M,N}^{(K)})^{\wedge n} \cong (\Sigma_n \wr H) \ltimes_{\Sigma_n \wr K} (\lambda_{n \times M, n \times N}^{(\Sigma_n \wr K)})$.

In order to finish the proof we have to show the following three things:

- (i) For every family $\{j_i : X_i \rightarrow Y_i\}_{i \in I}$ of generating $H^{\mathcal{U}}$ -projective (stable acyclic) cofibrations, the map $(\bigvee_I j_i)^{\wedge n}$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective (stable acyclic) cofibration.
- (ii) Given a pushout square

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ B & \longrightarrow & Y, \end{array}$$

where i is a wedge of generating $H^{\mathcal{U}}$ -projective (stable acyclic) cofibrations, the map $f^{\wedge n} : X^{\wedge n} \rightarrow Y^{\wedge n}$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective (stable acyclic) cofibration.

- (iii) Given a sequence $X_0 \rightarrow X_1 \rightarrow \dots$ of $H^{\mathcal{U}}$ -projective (stable acyclic) cofibrations, for which we already know that each individual map $(X_i)^{\wedge n} \rightarrow (X_{i+1})^{\wedge n}$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective (stable acyclic) cofibration, the map $(X_0)^{\wedge n} \rightarrow (\text{colim } X_i)^{\wedge n}$ is a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective (stable acyclic) cofibration.

This proves the proposition, because every $H^{\mathcal{U}}$ -projective (stable acyclic) cofibration is a retract of one obtained from the operations (i) – (iii).

Number (iii) is clear, because $(-)^{\wedge n}$ commutes with sequential colimits. Regarding (i), we use the distributive law recalled above to see that $(\bigvee_I j_i)^{\wedge n}$ decomposes as a wedge of maps of the form $(\Sigma_n \wr H) \ltimes_{\Sigma_{n_1} \wr H \times \dots \times \Sigma_{n_k} \wr H} (j_{a_1}^{\wedge n_1} \wedge \dots \wedge j_{a_k}^{\wedge n_k})$. We know that each factor is a (stable acyclic) $(n_i \times \mathcal{U})$ -projective cofibration of $(n_i \times \mathcal{U})$ -projective $(\Sigma_{n_i} \wr H)$ -symmetric spectra, so by Proposition 7.9 and Lemma 3.13 the induction of the smash product over them is a (stable acyclic) $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -projective cofibration. Hence, so is the wedge.

Finally, we prove (ii) by induction on n , using that (i) and (iii) are already dealt with. The case $n = 1$ is clear. So we fix an n larger than 1 and assume the statement being shown for all $k < n$. The $(\Sigma_n \wr H)$ -symmetric spectrum $Y^{\wedge n}$ is obtained from $X^{\wedge n}$ by inductively forming pushouts along the maps $(\Sigma_n \wr H) \ltimes_{\Sigma_k \wr H \times \Sigma_{n-k} \wr H} ((i^{\square k}) \wedge X^{\wedge(n-k)})$ for $k = 1, \dots, n$ (as explained, for example, in [SS12, Lemma A.8.]). Since the $i^{\square k}$ are $(\Sigma_k \wr H)^{k \times \mathcal{U}}$ -projective cofibrations if i is a wedge of generating cofibrations, as remarked above, and $X^{\wedge(n-k)}$ is $(\Sigma_{n-k} \wr H)^{(n-k) \times \mathcal{U}}$ -projective, the part about cofibrations again follows from Proposition 7.9. If i is a wedge of generating $H^{\mathcal{U}}$ -projective stable acyclic cofibrations, we have not shown that the pushout product is a $(\Sigma_k \wr H)^{k \times \mathcal{U}}$ -projective cofibration, but we can easily see that it is a $(\Sigma_k \wr H)$ -flat cofibration: Every $H^{\mathcal{U}}$ -projective stable acyclic cofibration of H -symmetric spectra of spaces is the geometric realization of an $H^{\mathcal{U}}$ -projective stable acyclic cofibration j of H -symmetric spectra of simplicial sets, with source and target $H^{\mathcal{U}}$ -projectively cofibrant, in particular of an H -flat cofibration of H -flat symmetric spectra of simplicial sets. But a map of equivariant symmetric spectra of simplicial sets is an equivariant flat cofibration if and only if its underlying map is a non-equivariant flat cofibration (Remark 4.12), so it follows from the monoidality of the non-equivariant flat model structure (Proposition 4.13) that the k -fold pushout product $j^{\square k}$ is a non-equivariant flat cofibration, hence a $(\Sigma_n \wr H)$ -flat cofibration and it follows that so is its geometric realization $i^{\square k}$. In particular, it is an h -cofibration. Therefore, it suffices to show that the quotients $(\Sigma_n \wr H) \ltimes_{\Sigma_k \wr H \times \Sigma_{n-k} \wr H} ((B/A)^{\wedge k} \wedge X^{\wedge(n-k)})$ are all $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stably contractible. Since we already know that each $X^{\wedge(n-k)}$ is $(\Sigma_{n-k} \wr H)^{(n-k) \times \mathcal{U}}$ -projective, in view of Proposition 7.9 it hence suffices to show that each $(B/A)^{\wedge k}$ is $(\Sigma_k \wr H)^{k \times \mathcal{U}}$ -stably contractible. For all $k < n$ this follows from the induction hypothesis. For $k = n$ we consider a different pushout where X is equal to A , Y is equal to B and the horizontal maps are identities. Again, we see that the map $i^{\wedge n} : A^{\wedge n} \rightarrow B^{\wedge n}$ is the composite of n maps, of which the first $n - 1$ are $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalences by the induction hypothesis. Since we have shown above that $i^{\wedge n}$ is also a $(\Sigma_n \wr H)^{n \times \mathcal{U}}$ -stable equivalence, this means that the last map in the composite must also be one. This last map is $i^{\square n}$ with quotient $(B/A)^{\wedge n}$, which finishes the proof. \square

Now we come to the flat case. Again, by Ken Brown's Lemma the following proposition implies Theorem 7.8.

Proposition 7.12. *Let $N \leq H \leq G$ be subgroup inclusions with N normal in G and $f : X \rightarrow Y$ an H -flat cofibration of H -flat H -symmetric spectra. Then $N_H^G f : N_H^G X \rightarrow N_H^G Y$ is a G -flat cofibration. If f is in addition an $H^{\mathcal{U}_H(N)}$ -stable equivalence, then $N_H^G f$ is a $G^{\mathcal{U}_G(N)}$ -stable equivalence.*

Proof. The method of proof is similar to that of Proposition 7.11 above. We explain the modifications needed and at which places we are making use of this specific type of H -set universe. We start with the generating (stable acyclic) flat cofibrations. The generating flat cofibrations are of the form $\mathcal{G}_m(i : A \rightarrow B)$ for i a genuine cofibration of $(G \times \Sigma_n)$ -CW complexes. Then $N_H^G(\mathcal{G}_m(i)) \cong \mathcal{G}_{G/H \times m}((\Sigma_{G/H \times m})_+ \wedge_{\Sigma_m^{G/H}} N_H^G i)$, which is a flat cofibration of G -symmetric spectra (as - by the same proof - is the map $(\mathcal{G}_m(i))^{\square N_H^G}$). Furthermore, as a consequence of Lemma A.3, this map is also a $G^{\mathcal{U}_G(N)}$ -level equivalence if $i : A \rightarrow B$ is in addition an $\mathcal{F}_{\mathcal{U}_H(N)}^{H, \Sigma_m}$ -equivalence, which proves the statement on the generating acyclic cofibrations of the level model structure. The additional generating acyclic cofibrations of the $H^{\mathcal{U}}$ -flat stable model structure are $H^{\mathcal{U}_H(N)}$ -projective cofibrations of $H^{\mathcal{U}_H(N)}$ -projective H -symmetric spectra, so Proposition 7.11 above implies that they are sent to $G^{G \ltimes_H \mathcal{U}_H(N)}$ -stable equivalences of $G^{G \ltimes_H \mathcal{U}_H(N)}$ -projective G -symmetric spectra under the norm. Since the G -set universe $G \ltimes_H \mathcal{U}_H(N)$ is a subuniverse of $\mathcal{U}_G(N)$, the change of universe for the projective model structures (cf. Section 6.1) implies that every $G^{G \ltimes_H \mathcal{U}_H(N)}$ -stable equivalence of $G^{G \ltimes_H \mathcal{U}_H(N)}$ -projective G -symmetric spectra is also a $G^{\mathcal{U}_G(N)}$ -stable equivalence.

We are then left to show the analogs of items (i), (ii) and (iii) in the proof of Proposition 7.11. Again, (iii) is easy. This time we show both items (i) and (ii) by induction over the order of G , the whole statement being trivial for $G = \{e\}$. So we assume the proposition already shown for all proper subgroups of G .

We start by proving (i): Given a wedge $\bigvee_I j_i : \bigvee_I X_i \rightarrow \bigvee_I Y_i$ of generating $H^\mathcal{U}$ -flat (stable acyclic) cofibrations, the distributive law shows that $N_H^G(\bigvee_I j_i)$ is a wedge of maps of the form $G \ltimes_K (\bigwedge_{i=1}^k N_{K \cap g_i H g_i^{-1}}^K c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H j_{f(g_i)})$, for a function $f : G/H \rightarrow I$. This immediately shows that $N_H^G(\bigvee_I j_i)$ is again a G -flat cofibration. If each j_i is a generating acyclic cofibration, we claim that every smash factor $N_{K \cap g_i H g_i^{-1}}^K c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H j_{f(g_i)}$ is a $K^{\mathcal{U}_K(K \cap N)}$ -stable equivalence. If K is equal to G , one can choose $g_i = 1$ and the term becomes simply $N_H^G(j_{f(1)})$, which we have shown to be a $G^{\mathcal{U}_G(N)}$ -stable equivalence. If K is a proper subgroup, $c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H j_{f(g_i)}$ is a $K \cap g_i H g_i^{-1}$ -flat cofibration and in addition a $(K \cap g_i H g_i^{-1})^{c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H \mathcal{U}_H(N)}$ -stable equivalence. But the $K \cap g_i H g_i^{-1}$ -set universe $c_{g_i}^* \text{res}_{g_i^{-1} K g_i \cap H}^H \mathcal{U}_H(N)$ is isomorphic to $\mathcal{U}_{K \cap g_i H g_i^{-1}}(K \cap N)$, and $K \cap N$ is a subgroup of $K \cap g_i H g_i^{-1}$ which is normal in K . So by the induction hypothesis, applying the norm $N_{K \cap g_i H g_i^{-1}}^K$ gives a $K^{\mathcal{U}_K(K \cap N)}$ -stable equivalence, as claimed. Hence, by the monoidality of the $K^{\mathcal{U}_K(K \cap N)}$ -flat stable model structure, the smash product over all these is also a $K^{\mathcal{U}_K(K \cap N)}$ -stable equivalence. Since $\mathcal{U}_K(K \cap N)$ is isomorphic to the restriction of $\mathcal{U}_G(N)$, we see that the induction to G is a $G^{\mathcal{U}_G(N)}$ -stable equivalence, so (i) is shown.

For item (ii) we take a pushout square as in the proof of Proposition 7.11 and again use that $Y^{\wedge n}$ can be obtained from $X^{\wedge n}$ by iteratively forming pushouts along the map $\Sigma_n \wr H \ltimes_{\Sigma_k \wr H \times \Sigma_{n-k} \wr H} (i^{\square k} \wedge X^{\wedge(n-k)})$ for $k = 1, \dots, n$. All these maps are $\Sigma_n \wr H$ -flat cofibrations by the same argument as in the proof of Proposition 7.11 and in particular G -flat cofibrations, proving the part on cofibrations. For the case where i is a wedge of generating $H^{\mathcal{U}_H(N)}$ -stably acyclic cofibrations, we consider the quotient $(\Sigma_n \wr H) \ltimes_{\Sigma_k \wr H \times \Sigma_{n-k} \wr H} (B/A)^{\wedge k} \wedge X^{\wedge(n-k)}$. It remains to show that when restricted along the chosen embedding $G \rightarrow \Sigma_n \wr H$, these quotients are all $G^{\mathcal{U}_G(N)}$ -stably contractible. This restriction is isomorphic to the wedge over the terms in the distributive law for $N_H^G(B/A \vee X)$ associated to those functions $f : G/H \rightarrow I$ which map exactly k elements to 1. For all but the constant function with value 1, the induction hypothesis and the same proof as for item (i) shows that the associated term is $G^{\mathcal{U}_G(N)}$ -stably contractible. Hence, it remains to show that $N_H^G(B/A)$ is $G^{\mathcal{U}_G(N)}$ -stably contractible. For this we again consider a different pushout with $X = A$, $Y = B$ and both horizontal maps the identities. We already know that the composite $N_H^G A \rightarrow N_H^G B$ is a $G^{\mathcal{U}_G(N)}$ -stable equivalence. Since we also showed that all maps but the last in the factorization are $G^{\mathcal{U}_G(N)}$ -stable equivalences, this means that the last one must also be one. This last map has quotient $N_H^G(B/A)$, which is hence $G^{\mathcal{U}_G(N)}$ -stably contractible, so we are done. \square

7.3 Model structures on commutative algebras

In this section we construct model structures on the category of commutative algebras over a fixed commutative G -symmetric ring spectrum R . The following theorem summarizes the results. We recall once more that for a normal subgroup N of G we denote by $\mathcal{U}_G(N)$ the full N -fixed G -set universe.

Theorem 7.13 (Model structures on commutative algebras). *Let R be a commutative G -symmetric ring spectrum and N be a normal subgroup of G . Then the positive $G^{\mathcal{U}_G(N)}$ -projective and the positive $G^{\mathcal{U}_G(N)}$ -flat model structures lift to the category of commutative R -algebras. In either of these categories, a map is a weak equivalence or fibration if and only if the underlying map is in the respective positive model structure on G -symmetric spectra.*

The cofibrations in the flat model structures do not depend on the universe and have the following convenient property (cf. [Shi04] for the non-equivariant version):

Proposition 7.14 (Convenience). *The underlying R -module map of a positive G -flat cofibration of commutative R -algebras $X \rightarrow Y$ is a positive G -flat cofibration of R -modules if X is (not necessarily positive) G -flat as an R -module. In particular, the G -symmetric spectrum underlying a positive G -flat commutative G -symmetric ring spectrum is positive G -flat.*

In the last section we discussed that given a subgroup inclusion $H \leq G$, the forgetful functor from the category of commutative G -symmetric ring spectra to the category of commutative H -symmetric ring spectra has a left adjoint, the norm N_H^G . Since fibrations and weak equivalences are created in underlying non-equivariant spectra, it follows directly from the change of group results of Section 6.2 that this becomes a Quillen adjunction for both the positive projective and the positive flat model structures. Moreover, the convenience property and Theorem 7.8 have the following consequence:

Corollary 7.15. *For every normal subgroup N of H the diagram*

$$\begin{array}{ccc} \mathrm{Ho}^{\mathcal{U}_H(N)}(\text{comm. } H\text{-rings}) & \xrightarrow{\mathbb{L}N_H^G} & \mathrm{Ho}^{\mathcal{U}_G(N)}(\text{comm. } G\text{-rings}) \\ \downarrow & & \downarrow \\ \mathrm{Ho}^{\mathcal{U}_H(N)}(HSp^\Sigma) & \xrightarrow{\mathbb{L}N_H^G} & \mathrm{Ho}^{\mathcal{U}_G(N)}(GSp^\Sigma) \end{array}$$

commutes up to natural isomorphism, where the vertical maps are forgetful functors.

Proof. We can compute the derived norm on the level of commutative ring spectra by replacing by a cofibrant commutative H -symmetric ring spectrum in the positive $H^{\mathcal{U}_H(N)}$ -flat model structure. By Proposition 7.14, the H -symmetric spectrum underlying a cofibrant commutative H -symmetric ring spectrum is (positive) H -flat and we know that the derived norm N_H^G on H -symmetric spectra can be computed by an H -flat replacement (Theorem 7.8), which finishes the proof. \square

In [HHR14], where they work with a projective model structure, this issue needs to be addressed differently (cf. [HHR14, Section B.8.]).

We now come to the proofs. In the non-equivariant version, a major input is the fact that given a positive flat symmetric spectrum X , the n -fold smash product $X^{\wedge n}$ is Σ_n -free, i.e., the map $(E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \rightarrow X^{\wedge n}/\Sigma_n$ is a stable equivalence. Equivariantly, the level of freeness depends on the chosen G -set universe. We recall that $E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n}$ denotes the universal space for the family of graph subgroups for homomorphisms $H \rightarrow \Sigma_n$ ($H \leq G$) with kernel containing $H \cap N$ and that $\mathcal{G}_m(-)$ stands for the semi-free G -symmetric spectrum functor in level m (cf. Section 2.7).

Lemma 7.16. *For every two natural numbers m, n with $m > 0$ and every cofibrant $(G \times \Sigma_m)$ -space A the map*

$$(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} (\mathcal{G}_m(A))^{\wedge n} \rightarrow (\mathcal{G}_m(A))^{\wedge n}/\Sigma_n$$

is a $G^{\mathcal{U}_G(N)}$ -level equivalence.

Remark 7.17. As remarked in [HHR14, B.120], there is an error in the analogous version of this result for G -orthogonal spectra in [MM02, Lemma III.8.4] (and also in the one for equivariant symmetric spectra in [Man04, Prop. 8.2]), where they always use an $E\Sigma_n$ with trivial G -action, even when working over non-trivial G -representation universes. In the case of G -orthogonal spectra and the complete universe it is corrected in [HHR14, Prop. B.117].

Proof. There is a natural $(G \times \Sigma_n)$ -isomorphism $(\mathcal{G}_m(A))^{\wedge n} \cong \mathcal{G}_{n \times m}((\Sigma_{n \times m})_+ \wedge_{\Sigma_m^n} A^{\wedge n})$ with diagonal G -action on A and Σ_n -action both on $A^{\wedge n}$ and by precomposition on $\Sigma_{n \times m}$ (and no Σ_n -action on the level $n \times m$, with respect to which the semi-free spectrum is formed). So the map above is naturally isomorphic to $\mathcal{G}_{n \times m}$ applied to the map

$$(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} (\Sigma_{n \times m})_+ \wedge_{\Sigma_m^n} A^{\wedge n} \rightarrow (\Sigma_{n \times m})_+ \wedge_{\Sigma_n \wr \Sigma_m} A^{\wedge n}$$

of cofibrant $(G \times \Sigma_{n \times m})$ -spaces. The isotropy of $\mathcal{U}_G(N)$ is closed under supergroups and so Lemma A.4 says that this map is an $\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_{n \times m}}$ -equivalence, which finishes the proof. \square

Proposition 7.18. *For every positive G -flat G -symmetric spectrum X the map*

$$(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} X^{\wedge n} \rightarrow X^{\wedge n} / \Sigma_n$$

is a $\underline{\pi}_^{\mathcal{U}_G(N)}$ -isomorphism and in particular a $G^{\mathcal{U}_G(N)}$ -stable equivalence.*

Proof. It suffices to prove the topological case. We explain the standard proof how to derive this from Lemma 7.16 (as, for example, in [EKMM97, Thm. III.5.1]). We prove by induction on k that the map $(E\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} (F^k X)^{\wedge n} \rightarrow (F^k X)^{\wedge n} / \Sigma_n$ is a $G^{\mathcal{U}}$ -stable equivalence for all $n \in \mathbb{N}$. Since X is positively G -flat, we have $F^0 X = *$ and so the induction start is trivial. We then fix a $k > 0$ and assume the statement being shown for $k - 1$ and all $n \in \mathbb{N}$. The k -th skeleton is obtained from the $(k - 1)$ st via a pushout along the map $\mathcal{G}_k(\nu_k(X)) : \mathcal{G}_k(L_k X) \rightarrow \mathcal{G}_k(X_k)$ for a genuine $G \times \Sigma_k$ -cofibration $\nu_k(X)$. Via the same decomposition as in the proof of Proposition 7.11 we see that $(F^k X)^{\wedge n}$ is obtained from $(F^{k-1} X)^{\wedge n}$ by inductively forming pushouts along the maps $\Sigma_n \rtimes_{\Sigma_j \times \Sigma_{n-j}} (\mathcal{G}_k(\nu_k(X)))^{\square j} \wedge (F^{k-1} X)^{\wedge(n-j)}$ for $1 \leq j \leq n$. These maps are G -flat cofibrations, so it suffices to show that for each quotient $\Sigma_n \rtimes_{\Sigma_k \times \Sigma_{n-k}} (\mathcal{G}_k(X_k/L_k X))^{\wedge j} \wedge (F^{k-1} X)^{\wedge(n-j)}$ the map

$$\begin{array}{c} (E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n})_+ \wedge_{\Sigma_j \times \Sigma_{n-j}} (\mathcal{G}_k(X_k/L_k X))^{\wedge j} \wedge (F^{k-1} X)^{\wedge(n-j)} \\ \downarrow \\ (\mathcal{G}_k(X_k/L_k X))^{\wedge j} \wedge (F^{k-1} X)^{\wedge(n-j)} / \Sigma_j \times \Sigma_{n-j} \end{array}$$

is a $\underline{\pi}_*^{\mathcal{U}_G(N)}$ -isomorphism. Since $E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n}$ is $\Sigma_j \times \Sigma_{n-j}$ -homotopy equivalent to $E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_j} \times E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_{n-j}}$, this map is the smash product of the two maps

$$(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_j})_+ \wedge_{\Sigma_j} (\mathcal{G}_k(X_k/L_k X))^{\wedge j} \rightarrow (\mathcal{G}_k(X_k/L_k X))^{\wedge j} / \Sigma_j$$

and

$$(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_{n-j}})_+ \wedge_{\Sigma_{n-j}} (F^{k-1} X)^{\wedge(n-j)} \rightarrow (F^{k-1} X)^{\wedge(n-j)} / \Sigma_{n-j}.$$

The first is a $\underline{\pi}_*^{\mathcal{U}_G(N)}$ -isomorphism by Lemma 7.16, the latter by the induction hypothesis. Since all the spectra involved are G -flat, monoidality of the flat stable model structure implies that their smash product is also a $G^{\mathcal{U}}$ -stable equivalence and the induction step is proved. The statement about X itself then follows by passing to the colimit. \square

For the homotopical properties of $(-)^{\wedge n} / \Sigma_n$ we hence need to examine those of the functor $(E\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} X^{\wedge n}$.

Lemma 7.19. *Let \mathcal{U} be a G -set universe, Y a $(G \times \Sigma_n)$ -symmetric spectrum which is $K^{\mathcal{U}}$ -contractible for all graph subgroups K in $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$ (where K acts on \mathcal{U} through its projection to G), and A a cofibrant $(G \times \Sigma_n)$ -space set with isotropy in $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n}$. Then $A \wedge_{\Sigma_n} Y$ is $G^{\mathcal{U}}$ -stably contractible.*

Proof. We let A be an I_G -cell complex, the general statement follows since $G^{\mathcal{U}}$ -stable equivalences are closed under retracts. By an induction over the cells it suffices to show that $(G \times \Sigma_n)/K_+ \wedge_{\Sigma_n} Y$ is $G^{\mathcal{U}}$ -stably contractible for all K in $\mathcal{F}_U^{G, \Sigma_n}$. Let H be the projection of K to G . Then $(G \times \Sigma_n)/K_+ \wedge_{\Sigma_n} Y$ is G -isomorphic to $G \rtimes_H \text{res}_H^K Y$, which is $G^{\mathcal{U}}$ -stably contractible by the assumption and the fact that induction maps $H^{\mathcal{U}}$ -stable equivalences to $G^{\mathcal{U}}$ -stable equivalences. \square

To establish the model structures we use results of [Whi14]. The following is the main input for applying them:

Proposition 7.20. *Let $i : A \rightarrow B$ be a map of G -symmetric spectra. Then:*

- (i) *If i is a (positive) G -flat cofibration, then $i^{\square n}/\Sigma_n$ is again a (positive) G -flat cofibration.*
- (ii) *If i is a positive G -flat cofibration and $G^{\mathcal{U}_G(N)}$ -stable equivalence, then $i^{\square n}/\Sigma_n$ is a $G^{\mathcal{U}_G(N)}$ -stable equivalence.*

Proof. By [Whi14, Lemma 5.1], it suffices to show (i) on the class of generating G -flat cofibrations, which are all of the form $\mathcal{G}_m(j) : \mathcal{G}_m(A) \rightarrow \mathcal{G}_m(B)$ for a genuine $(G \times \Sigma_m)$ -cofibration j and $m > 0$. But then $(\mathcal{G}_m(j))^{\square n}$ is equal to $\mathcal{G}_{n \times m}(\Sigma_{n \times m} \rtimes_{\Sigma_m} j^{\square n})$, which is even a $\Sigma_n \wr G$ -flat cofibration because $\Sigma_{n \times m} \rtimes_{\Sigma_m} j^{\square n}$ is a genuine $\Sigma_n \wr G$ -cofibration. This proves that $(-)^{\square n}/\Sigma_n$ preserves (positive) G -flat cofibrations.

For (ii) we use that we already know that $i^{\square n}/\Sigma_n$ is a G -flat cofibration, and hence it suffices to show that the quotient $(B/A)^{\wedge n}/\Sigma_n$ is $G^{\mathcal{U}}$ -stably contractible. Since B/A is positive G -flat, we know by Proposition 7.18 that the quotient is $G^{\mathcal{U}_G(N)}$ -stably equivalent to $(E\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} (B/A)^{\wedge n}$. We now want to apply Lemma 7.19 and have to check that for every subgroup H of G and every homomorphism $\alpha : H \rightarrow \Sigma_n$ with kernel containing $H \cap N$ the spectrum $(B/A)^{\wedge n}$ is $H^{\mathcal{U}_H(H \cap N)}$ -stably contractible, where the H -action is via the graph embedding $H \rightarrow G \times \Sigma_n$. But $(B/A)^{\wedge n}$ is H -isomorphic to a smash product of $N_{K_i}^H(B/A)$ for subgroups K_i of H which contain $H \cap N$. The G -symmetric spectrum B/A is $G^{\mathcal{U}_G(N)}$ -stably contractible, hence it is also $K_i^{\mathcal{U}_{K_i}(K_i \cap N)}$ -stably contractible. Since K_i contains $H \cap N$, we see that $K_i \cap N$ equals $H \cap N$, which is normal in H . So we can use Theorem 7.8 to deduce that each $N_{K_i}^H(B/A)$ is $H^{\mathcal{U}_H(H \cap N)}$ -stably contractible. All of them are also H -flat (by Proposition 7.11), hence their smash product is $H^{\mathcal{U}_H(H \cap N)}$ -stably contractible and we are done. \square

In the language of [Whi14], this implies that the positive $G^{\mathcal{U}_G(N)}$ -flat stable model structure satisfies the strong commutative monoid axiom ([Whi14, Def. 3.4.]), while the positive $G^{\mathcal{U}_G(N)}$ -projective stable model structure satisfies the weak commutative monoid axiom ([Whi14, Rem. 3.3]).

Proof of Theorem 7.13. This follows from Proposition 7.20 via [Whi14, Thm 3.2.] in the flat case and [Whi14, Rem. 3.3.] in the projective case. \square

Proof of Proposition 7.14. If one assumes that the source is a positive G -flat R -module, this is [Whi14, Prop. 3.5.]. However, the only place where positive G -flatness of the source is used in the proof is to ensure that smashing with it preserves positive G -flat cofibrations of R -modules. For this it is sufficient that it is G -flat as an R -module. \square

A Appendix

In this appendix we list several technical lemmas about the interplay of the families $\mathcal{F}_U^{G, \Sigma_n}$ for varying n and G , which we left out of the main text to shorten the exposition. The proofs are all similar in style and rely on the following:

Lemma A.1. *Let G be a discrete group with a normal subgroup K , X a cofibrant G -space on which the normal subgroup K acts freely away from the basepoint and H a subgroup of the quotient G/K . Then there is a natural homeomorphism*

$$(X/K)^H \cong \bigvee_{[\alpha: H \rightarrow G]} X^{\text{im}(\alpha)} / (C(\text{im}(\alpha)) \cap K), \quad (\text{A.1})$$

where the wedge is taken over a system of representatives of K -conjugacy classes of group homomorphisms $\alpha : H \rightarrow G$ lifting the inclusion $H \rightarrow G/K$, and $C(\text{im}(\alpha))$ denotes the subgroup of elements commuting with the image of α .

Sketch of proof. Let $\alpha : H \rightarrow G$ be such a group homomorphism and $[h] \in H$ an element represented by $h \in G$. Then $h \cdot \alpha([h])^{-1}$ maps to e under the projection, hence it lies in K . So if x is an $\text{im}(\alpha)$ -fixed point of X , then we have $[h][x] = [hx] = [h\alpha(h)^{-1}x] = [x]$, so $[x]$ is an H -fixed point in the quotient. This shows that for each such α the composition $X^{\text{im}(\alpha)} \rightarrow X \rightarrow X/K$ lands in the H -fixed points and we obtain a map from the wedge of the $X^{\text{im}(\alpha)}$ over all such α to $(X/K)^H$. Furthermore, two such summands only intersect in the basepoint, because the K -action is free. The normal subgroup K acts on this wedge, an element k sends the subspace $X^{\text{im}(\alpha)}$ homeomorphically onto $X^{\text{im}(k\alpha k^{-1})}$ and so we obtain an injective map from the right side of the expression A.1 to the left. On the other hand, given an element $[x]$ fixed by H and represented by $x \in X$, there is an associated homomorphism $H \rightarrow G$ sending $[h]$ to $k(h)^{-1} \cdot h$, where $k(h)$ is the unique element in K which satisfies $hx = k(h)x$. It is an easy check that this is indeed independent of the chosen representative h , that it defines a group homomorphism lifting the inclusion $H \rightarrow G/K$ and that x is an $\text{im}(\alpha)$ -fixed point. Hence, the map is also surjective and that it is a homeomorphism follows from X being cofibrant. \square

Lemma A.2. *Let \mathcal{U} be a G -set universe. Then for every $\mathcal{F}_U^{G, \Sigma_m}$ -equivalence between cofibrant $(G \times \Sigma_m)$ -spaces $f : X \rightarrow Y$ and every cofibrant $(G \times \Sigma_k)$ -space A , the map $\Sigma_{m+k} \ltimes_{\Sigma_m \times \Sigma_k} (f \wedge A)$ is an $\mathcal{F}_U^{G, \Sigma_{m+k}}$ -equivalence.*

Proof. We let L be a subgroup contained in $\mathcal{F}_U^{G, \Sigma_{m+k}}$. By Lemma A.1 above (applied to the normal subgroup $\Sigma_m \times \Sigma_k$ of $\Sigma_{m+k} \times \Sigma_m \times \Sigma_k$), the L -fixed points of $\Sigma_{m+k} \ltimes_{\Sigma_m \times \Sigma_k} (X \wedge A)$ are naturally isomorphic to a wedge of terms of the form $((\Sigma_{m+k})_+ \wedge X \wedge A)^{\Gamma_\alpha} / C(\text{im}(\alpha))$, where α is a homomorphism $L \rightarrow \Sigma_m \times \Sigma_k$, and it suffices to show that $((\Sigma_{m+k})_+ \wedge f \wedge A)^{\Gamma_\alpha} / C(\text{im}(\alpha))$ is a weak equivalence for every such α . We show that $((\Sigma_{m+k})_+ \wedge f \wedge A)^{\Gamma_\alpha}$ is a weak equivalence. The action of $C(\alpha)$ is free, so this gives the statement. Since L is contained in $\mathcal{F}_U^{G, \Sigma_{m+k}}$, it is the graph of a homomorphism $\beta : H \rightarrow \Sigma_{m+k}$ for a subgroup H of G and for which the associated H -set $\underline{m+k}_\beta$ embeds into \mathcal{U} . Composing with α gives two further homomorphisms $\alpha_1 : H \rightarrow \Sigma_m$ and $\alpha_2 : H \rightarrow \Sigma_k$ with associated H -sets \underline{m}_{α_1} and \underline{k}_{α_2} . Hence, the Γ_α -fixed points of Σ_{m+k} are given by the set of H -isomorphisms from $\underline{m}_{\alpha_1} \sqcup \underline{k}_{\alpha_2}$ to $\underline{m+k}_\beta$. If \underline{m}_{α_1} does not embed into \mathcal{U} , then there cannot exist such an isomorphism, in which case both sides are trivial and hence the map is necessarily a weak equivalence. If \underline{m}_{α_1} does embed into \mathcal{U} , then f induces an equivalence between the Γ_{α_1} -fixed points of X and Y , since these are given by the fixed points of the graph of α_1 and f is an $\mathcal{F}_U^{G, \Sigma_m}$ -equivalence. Hence, the whole term is a weak equivalence and we are done. \square

The next lemma deals with the multiplicative norm $N_H^G X$ for a based H -space X and a subgroup inclusion $H \leq G$. We will make use of the double coset formula

$$\text{res}_K^G N_H^G X \cong \bigwedge_{[g] \in K \backslash G / H} N_{gHg^{-1} \cap K}^K c_g^* \text{res}_{H \cap g^{-1}Kg}^H X.$$

Provided that X also has a Σ_m -action commuting with the H -action, the norm $N_H^G X$ has a $G \ltimes (\Sigma_m)^{G/H}$ -action, with G acting on $(\Sigma_m)^{G/H}$ via its action on G/H .

The H -universe we use is the full N -fixed one $\mathcal{U}_H(N)$ for a subgroup N of H which is normal in G .

Lemma A.3. *Let $n, m \in \mathbb{N}$ and $j : A \rightarrow B$ an $\mathcal{F}_{\mathcal{U}_H(N)}^{H, \Sigma_m}$ -equivalence of cofibrant $(H \times \Sigma_m)$ -spaces. Then $(\Sigma_{G/H \times m})_+ \wedge_{\Sigma_m^{G/H}} N_H^G j$ is an $\mathcal{F}_{\mathcal{U}_G(N)}^{G, \Sigma_{G/H \times m}}$ -equivalence. Here, the G -action is both through $N_H^G(-)$ and by precomposition with the inverse on $\Sigma_{G/H \times m}$.*

Proof. Again we have to show that for every homomorphism $\beta : J \rightarrow \Sigma_{G/H \times m}$ from a subgroup J of G the Γ_β -fixed points of $(\Sigma_{G/H \times m})_+ \wedge_{\Sigma_m^{G/H}} N_H^G j$ are a weak equivalence, provided that the associated J -action on $G/H \times \underline{m}$ embeds into $\mathcal{U}_J(J \cap N)$, i.e., has $J \cap N$ in the kernel. Using Lemma A.1 above for the normal subgroup $\Sigma_m^{G/H}$ of $\Sigma_{G/H \times m} \times (G \ltimes \Sigma_m^{G/H})$, these fixed points naturally decompose into a wedge with summands of the form $((\Sigma_{G/H \times m})_+ \wedge N_H^G j)^{\Gamma_\alpha} / (C(\text{im}(\alpha)) \cap \Sigma_m^{G/H})$, where α is a group homomorphism $J \rightarrow G \ltimes (\Sigma_m^{G/H})$ lifting the inclusion $J \rightarrow G$. We again consider the map on fixed points before quotienting out by $C(\text{im}(\alpha)) \cap \Sigma_m^{G/H}$ and show that it is a weak equivalence. Since $G \ltimes \Sigma_m^{G/H}$ comes with an action on $G/H \times m$, the map α also induces another action of J , which we denote by $(G/H \times m)_\alpha$. Let g_1, \dots, g_k be a system of double coset representatives for $J \backslash G/H$ and $J_i = J \cap g_i H g_i^{-1}$ the J -stabilizer of $g_i H$ in G/H , in particular $g_i H \times m$ becomes a J_i -set with the restricted action through α . This gives us two things: A J -decomposition

$$(G/H \times m)_\alpha \cong \bigsqcup_{i=1, \dots, k} J \ltimes_{J_i} (g_i H \times m)_{\alpha|_{J_i}}$$

and an isomorphism

$$((\Sigma_{G/H \times m})_+ \wedge_{\Sigma_m^{G/H}} N_H^G j)^{\text{im}(\alpha)} \cong \text{Iso}_J((G/H \times m)_\beta, (G/H \times m)_\alpha)_+ \wedge \bigwedge_{i=1, \dots, k} (N_{J_i}^J \text{res}_{J_i}^{H \times \Sigma_m} A)^J$$

where the last restriction is formed along the composition

$$J_i \xrightarrow{(incl, \alpha)} g_i H g_i^{-1} \times \Sigma_m \xrightarrow{c_g^* \times \Sigma_m} H \times \Sigma_m.$$

The J -fixed points of a norm $N_{J_i}^J$ are isomorphic to the J_i -fixed points of the argument, so the last smash product factor becomes $(\text{res}_{J_i}^{H \times \Sigma_m} A)^{J_i}$. Now, if $L \cap N$ acts non-trivially on $(G/H \times m)_\alpha$, the latter cannot be isomorphic to $(G/H \times m)_\beta$ and hence in that case the fixed points are trivial on both sides and the statement is shown. On the other hand, if $J \cap N$ does act trivially on $(G/H \times m)_\alpha$, then each $J_i \cap N$ acts trivially on $(g_i H \times m)_{\alpha|_{J_i}}$. Hence the kernel of $\alpha|_{J_i}$ contains $J_i \cap N$ and so j induces a weak equivalence on $(\text{res}_{J_i}^{H \times \Sigma_m} A)^{J_i}$. Thus, in that case $((\Sigma_{G/H \times m})_+ \wedge N_H^G j)^{\Gamma_\alpha}$ is a weak equivalence as the smash product of weak equivalences between cofibrant spaces, which finishes the proof. \square

In the following lemma we need the technical hypothesis that the isotropy of the G -set universe \mathcal{U} is closed under supergroups, i.e., that whenever an orbit G/K embeds into \mathcal{U} , so does G/L for all $K \leq L \leq G$. We note that if \mathcal{U} has this property, then so do all restrictions of \mathcal{U} to a subgroup of G .

Lemma A.4. *Let A be a cofibrant $(G \times \Sigma_m)$ -space for $m > 0$, n a natural number and \mathcal{U} a G -set universe whose isotropy is closed under supergroups. Then the map*

$$(E\mathcal{F}_{\mathcal{U}}^{G, \Sigma_n})_+ \wedge_{\Sigma_n} (\Sigma_{n \times m})_+ \wedge_{\Sigma_m^n} A^{\wedge n} \rightarrow (\Sigma_{n \times m})_+ \wedge_{\Sigma_n \wr \Sigma_m} A^{\wedge n}$$

is an $\mathcal{F}_{\mathcal{U}}^{G, \Sigma_{n \times m}}$ -equivalence. Here, the Σ_n -action on $(\Sigma_{n \times m})_+ \wedge_{\Sigma_m^n} A^{\wedge n}$ is both through A^n and by precomposition on $\Sigma_{n \times m}$, permuting the blocks of size m . G only acts through $A^{\wedge n}$.

Proof. We have to show that it induces an equivalence on all fixed points for graph subgroups $L \in \mathcal{F}_U^{G, \Sigma_{n \times m}}$. Since $m > 0$, the map $\Sigma_n \wr \Sigma_m \rightarrow \Sigma_{n \times m}$ is injective and so the wreath product acts freely on both sides and we can apply Lemma A.1 above for the direct product $(G \times \Sigma_{m \times n}) \times (\Sigma_n \wr \Sigma_m)$ with normal subgroup $\Sigma_n \wr \Sigma_m$. So we have to show that the map $(E\mathcal{F}_U^{G, \Sigma_n})_+ \wedge (\Sigma_{n \times m})_+ \wedge A^{\wedge n} \rightarrow (\Sigma_{n \times m})_+ \wedge A^{\wedge n}$ induces an equivalence on graph subgroups in $H \times \Sigma_{n \times m} \times \Sigma_n \wr \Sigma_m$ (where H is the projection of L to G) for homomorphisms (α, β) of which $\underline{n} + m_\alpha$ allows an H -embedding into \mathcal{U} . The wreath product $\Sigma_n \wr \Sigma_m$ acts on $E\mathcal{F}_U^{G, \Sigma_n}$ through the projection to Σ_n , and hence the fixed points of such a graph subgroup on $E\mathcal{F}_U^{G, \Sigma_n}$ are contractible if β followed by this projection to Σ_n defines an H -set structure on \underline{n} which embeds into \mathcal{U} , and empty otherwise. Hence, we have to show that if this H -action on \underline{n} does not embed into \mathcal{U} , then the fixed points on the right hand side are also empty. We claim that already the fixed points on $\Sigma_{n \times m}$ are empty. As in the proofs before, a bijection in $\Sigma_{n \times m}$ is fixed under this graph subgroup if and only if it is an isomorphism from the H -set structure on $\underline{n} \times \underline{m}$ via β to the one via α . Since we assumed that α allows an embedding into \mathcal{U} , we thus have to show that if $\underline{n}_{\text{pro}\beta}$ is not H -embeddable, then neither is $\underline{n} \times \underline{m}_\beta$. We decompose the H -action on Σ_n into orbits N_1, \dots, N_k , choose elements $n_i \in N_i$ and let $H_i \leq H$ be the stabilizer of n_i . Then there is an H -isomorphism

$$(\underline{n} \times \underline{m})_\beta \cong \bigsqcup_{i=1, \dots, k} H \ltimes_{H_i} (n_i \times m).$$

By assumption, for one of the H_i 's the H -set H/H_i does not embed into \mathcal{U} . So, since the isotropy of \mathcal{U} is closed under supergroups, $H \ltimes_{H_i} (n_i \times m)$ also does not H -embed into \mathcal{U} (the isotropy of an element can only be smaller) and hence neither does $(\underline{n} \times \underline{m})_\beta$, and so we are done. \square

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